# Permutation Models and SVC

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**Abstract**

Let $M$ be a model of ZFAC (ZFC modified to allow a set of atoms), and let $N$ be an inner model with the same set of atoms and the same pure sets (sets with no atoms in their transitive closure) as $M$. We show that $N$ is a permutation submodel of $M$ if and only if $N$ satisfies the principle SVC (Small Violations of Choice), a weak form of the axiom of choice which says that in some sense, all violations of choice are localized in a set. A special case is considered in which there exists an SVC witness which satisfies a certain homogeneity condition.

## 1 Introduction and Main Result

The principle SVC (Small Violations of Choice) is a weak form of AC (Axiom of Choice), introduced by Blass [1], which says that all failures of AC are localized in a set $S$:

> SVC: There is a set $S$ such that, for every set $a$, there is an ordinal $\alpha$ and a function from $S \times \alpha$ onto $a$.

When $S$ is such a set, we say that "SVC holds with $S$" and that $S$ is an *SVC witness*.

The main new result of this paper can be stated as follows: Let $M$ be a model of ZFA (ZF modified to allow a set atoms) in which AC holds, and let $N$ be an inner model which has the same pure part and same set of atoms as $M$. If $N \models$ SVC, then $N$ is a permutation submodel of $M$.

**Definitions and Conventions.** The theory *ZFA* is a modification of ZF allowing atoms, also known as urelements. See Jech [4] for a precise definition. A model of ZFA may have a proper class of atoms; however, for this paper we redefine ZFA to include an axiom which says that the class of atoms is a set (always denoted by $A$).

Similarly, proper class forcing will not be considered in this paper; by *forcing* or *generic extension* it is to be understood that only a set of forcing conditions is permitted.

In a model of ZFA, a *pure set* is a set with no atoms in its transitive closure, and the *pure part* or *kernel* is the class of all pure sets; the pure part is a model of ZF.

Our definition of *permutation submodel* will be almost the same as that given in [4] (or see Jech [5] for more detail). A permutation model is determined by a model $M$ of ZFAC, a group $G$ of permutations of the set $A$ of atoms, and a normal filter $\mathcal{F}$ on $G$, Typically, it is assumed that $G$ is in $M$. In this paper, we only require that $G$ be in some generic extension of $M$ by a cardinal collapse (or by any almost homogeneous notion of forcing); the development of the basic theory is nearly unchanged. (See Hall [3] for an example of some $N \subset M$ where $N$ is a permutation submodel of $M$ which cannot be obtained by a group $G$ in $M$.)

The definition of permutation model above is needed so that the statement of the theorem below is correct. However, from this point we will not work much with the definition of permutation model. Instead, we build on and generalize results in the paper [3], which gives a characterization of permutation submodels in terms of forcing. The new result stated above will be treated as part of the following main theorem.

**Theorem 1.** *Let $M$ be a transitive model of ZFAC, and let $N \subseteq M$ be transitive submodel of ZFA such that $N$ and $M$ have the same set of atoms and the same pure part. The following are equivalent:*

*(a) $N$ is a permutation submodel of $M$.*

*(b) $M$ is a generic extension of $N$.*

*(c) $N$ satisfies SVC.*

The equivalence between (a) and (b) is Theorem 4.1(a) of [3]. The implication from (a) to (c) is Theorem 4.2 of [1] (no serious changes are required to make the proof work for our slightly generalized permutation models). The implication from (b) to (c) follows immediately from Theorem 4.6 of [1], stated here (generalized slightly to allow a set of atoms):

**Theorem 2.** *A model of ZFA satisfies SVC if and only if some generic extension satisfies AC.*

*Hint:* If $\mathbb{P}$ is a notion of forcing such that $\Vdash_{\mathbb{P}}$ AC, then SVC holds with $\mathbb{P}$. □

The proof of Theorem 1 can therefore be completed by proving that (c) implies either (a) or (b). We will prove, in section 3, that (c) implies (b); we have not found a nice proof that (c) implies (a) without in effect going through (b).

# 2 Questions

Let ZFAC$^K$ be the theory of ZFA (with a set of atoms) + "AC for pure sets." It was claimed in passing on the first page of [3] that SVC is a theorem of ZFAC$^K$. The claim was mistaken. No proof is known; also no disproof is known.

Consider a weaker version of the claim. If $N \models$ ZFAC$^K$ and $M$ is an extension with the same pure part and set of atoms as $N$ such that $M \models$ ZFAC, then we'll say that $M$ is a *choice extension* of $N$. By Theorem 1, the following questions are equivalent.

**Question 3.** (a) For a given model of ZFAC$^K$, is every choice extension a generic extension? (b) Does SVC hold in every model of ZFAC$^K$ that has a choice extension?

If the answer is "yes," then the three equivalent conditions of Theorem 1 are simply true under the given hypotheses.

## 3 Proof of the Main Theorem

The following lemma is proved in Blass and Scedrov [2]; a sketch of the proof is included here because it contains an idea to be used later.

**Lemma 4.** *Let $M$ be a model of ZFAC with pure part $K$, and let $f : A' \to A$ be a bijection from a pure set to the set of atoms. Then $M$ is the smallest model of ZFAC which contains $K$ and $f$.*

*Proof (sketch).* For convenience, assume that all the elements of $A'$ have the same rank, and let $X_0 \notin A'$ also be a pure set of that rank. As in the proof of Lemma 15.47 in [4], construct a model $M'$ of ZFAC inside $K$ whose set of atoms is $A'$. The elements of $M'$ are obtained by iterating the power set operation over $A'$, modified by letting $X_0$ stand in for the empty set each time. Now $\langle M', \in \rangle$ is a model of ZFAC.

There is a unique collapsing map from $M'$ onto $M$ whose restriction to $A'$ is $f$. This map is $\Delta_1$-definable using $f$ as a parameter, so $M$ is generated by $K$ and $f$. □

Note that the collapsing map $M' \to M$ in the proof above is an isomorphism.

To prove (c) implies (b) in Theorem 1, we start with a model $M$ of ZFAC with a submodel $N$ as in the hypotheses. As in the proof of Lemma 4, let $M'$ be a copy of $M$ contained in the pure part of $M$ (which is also the pure part of $N$), with $A'$ as its set of atoms. There is a copy $N'$ of $N$ contained in $M'$. In $N$, the set $A$ of atoms is not well-orderable (excepting the boring case $N = M$), and $N$ does not see that $N$ is isomorphic to $N'$ or to any other submodel of $M'$. We'll build a notion of forcing in $N$ out of certain partial embeddings from $N$ to $N'$.

Of all generic extensions of $N$ which add a well-ordering of $A$, $M$ is a minimal such model. (Other extensions which add the same well-orderings of $A$ that $M$ has must contain $M$ and also add new pure sets.) Intuitively, to get a "small" extension like $M$ generically, we want a notion of forcing whose conditions are as large as possible; perhaps a proper class containing arbitrarily large partial embeddings $N \rightharpoonup N'$. The assumption that SVC holds in $N$ turns out to ensure that a mere set of forcing conditions suffices, and also ensures, by way of the next lemma, that the dense subsets will be well-behaved.

In the following Lemma 5, think of $S$ as an SVC witness in a model of ZFA. Some form of this lemma was first pointed out to me by Omar De la Cruz.

**Lemma 5.** *If $f : S \times \alpha \to B$ is onto, then for every $D \subseteq B$, there is a pure set $y$ and a well-ordering $x$ of a subset of $\mathscr{P}(S)$ such that $D$ is $\Delta_0$-definable from the parameters $f$, $x$, and $y$.*

*Proof.* Let $D \subseteq B$, and consider the set $f^{-1}[D] \subseteq S \times \alpha$. $D$ is $\Delta_0$-definable from $f$ and $f^{-1}[D]$; it remains to show that $f^{-1}[D]$ is $\Delta_0$-definable from some $x$ and $y$ as in the statement of the lemma. Define a one-to-one partial function $b : \mathscr{P}(S) \rightharpoonup \mathscr{P}(\alpha)$ by

$$b(T) = \begin{cases} \{\, \beta < \alpha \mid f^{-1}[D] \cap (S \times \{\beta\}) = T \times \{\beta\} \,\} & \text{if this is nonempty,} \\ \text{undefined} & \text{else.} \end{cases}$$

Observe that $\mathsf{Ran}(b)$ is a pairwise disjoint set of sets of ordinals, and hence is a well-orderable pure set. Let $y$ be a well-ordering of $\mathsf{Ran}(b)$, and let $x$ be the corresponding well-ordering of $\mathsf{Dom}(b)$. Then $b$ is $\Delta_0$-definable from $x$ and $y$, and $f^{-1}[D]$ is in turn $\Delta_0$-definable from $b$. □

*Proof of Theorem 1, (c) implies (b).* Let $M$ be a transitive model of ZFAC and let $N$ be an inner model of ZFA, both with the same kernel $K$ and set of atoms $A$. As in the discussion above, let $M' \subset K$ be an isomorphic copy of $M$, with $A'$ as its set of atoms. Any bijection $A \to A'$ in $M$ can be extended uniquely to an isomorphism $M \to M'$. Let $j : M \to M'$ be such an isomorphism, and for $x \in M$ we'll write $x' = j(x)$, and $N' = j[N]$. Observe that if $j_1 : M \to M'$ is any other isomorphism, then $j$ and $j_1$ agree on $K$, since there is only one isomorphism $K \to K'$.

For a function $p$ whose range is contained in $M'$, define a new function $\tilde{p} : \mathsf{Ran}(p) \to K'$ by $\tilde{p}(r) = r'$; in other words, $\tilde{p} = j \restriction \mathsf{Ran}(p)$. It is immediate from the definition that if $p$ and $q$ are any two functions with the same range, then $\tilde{p} = \tilde{q}$. The remainder of this paragraph is optional, for readers interested in the motivation for defining $\tilde{p}$. Suppose that $p$ can be extended to an isomorphism $N \to N'$, and consider the function $p^+ = \bigcap\{\, i \supset p \mid i : N \to N' \text{ is an isomorphism} \,\}$, the intersection of all isomorphisms $N \to N'$ which extend $p$. Think of the domain of $p^+$ as the *extended domain* of $p$. For example, if $x \in \mathsf{Dom}\, p$, then $\{x\}$ is certainly in the extended domain of $p$. Each pure set $y$ is also in the extended domain (since all isomorphisms $N \to N'$ agree on pure sets); $p^+(y) = y'$. It turns out that $p$ is in its own extended domain; to show this, it suffices to show that each $\langle x, y\rangle \in p$ is in the extended domain of $p$. To this end, let $i : N \to N'$ be any isomorphism extending $p$. Clearly $i(x) = p(x) = y$. Since $y \in \mathsf{Ran}(p)$ is a pure set, $i(y) = y'$. Thus $i(\langle x, y\rangle) = \langle y, y'\rangle$ for any isomorphism $i$ extending $p$. It follows that $p \in \mathsf{Dom}(p^+)$, and $p^+(p) = \tilde{p}$.

Suppose $N$ satisfies SVC with $S$. Working in $N$, we will now define a notion of forcing $\mathbb{P}$. Let $T = \mathscr{P}(S)$. Let $F$ be the set of all functions from subsets of $T$ to $T'$; fix an ordinal $\alpha$ and a surjection $f : S \times \alpha \to F$. Note that although the priming function $j$ is not in $N$, the restriction $j \restriction K$ is in $N$ (it is the unique isomorphism $K \to K'$), so we may freely apply primes to pure sets. It follows that the tilde operation $p \mapsto \tilde{p}$ also makes sense in $N$ (when applied to functions whose ranges are pure sets). We will also refer to the two particular sets $T'$ and $f'$. Finally, the definition of $\mathbb{P}$ will also use the term $N'$. To avoid the implicit assumption that $N'$ is a definable class in $N$, one could replace $N'$ in the definition of $\mathbb{P}$ with some sufficiently large initial segment $N'_\xi$.

Let $\mathbb{P}$ be the set of all partial injections $p : T \rightharpoonup T'$ such that $\mathsf{Ran}(p)$ is well-orderable in $N'$, and for every $\Delta_0$ formula $\phi$, every $y \in K$, and every (transfinite) sequence $\mathbf{x}$ of elements of $\mathsf{Dom}\, p$, we have $p(\mathbf{x}) \in N'$ and

$$N \models \phi(\mathbf{x}, p, y, f) \quad \leftrightarrow \quad N' \models \phi(p(\mathbf{x}), \tilde{p}, y', f'). \tag{$*$}$$

The domain of each $p \in \mathbb{P}$ is well-orderable (since $\mathsf{Ran}(p) \subset T'$ is always a pure set). Conversely, if $X \subset T$ is well-orderable, then $X$ is the domain of the function $p = j \restriction X \in \mathbb{P}$. To see that $p \in N$, let $k_1 : X \to \kappa$ be a bijection in $N$ from $X$ to some ordinal $\kappa$. Since $j \in M$, there is clearly a $k_2 : \kappa \to \mathsf{Ran}(p)$ such that $p = k_2 \circ k_1$. But this $k_2$ would be a pure set, so $k_2$ and hence $p$ are in $N$.

Now, back out in $M$, define $G = \{\, p \in \mathbb{P} \mid p \subset j \,\}$. It is not hard to see that $G$ is a filter in $\mathbb{P}$. It remains to show that $G$ is generic over $N$. This will suffice because $M \subseteq N[G]$ by Lemma 4, and since $N \subset M$ and $G \in M$ it must be that $M = N[G]$.

Toward showing that $G$ is $\mathbb{P}$-generic over $N$, let $D \in N$ be a dense subset of $\mathbb{P}$. Applying Lemma 5, we get a parameter $y \in K$, a parameter $\mathbf{x}$ which we can think of as a sequence of elements in $T$, and a $\Delta_0$ formula $\varphi$ such that for all $t$,

$$t \in D \quad \leftrightarrow \quad \varphi(\mathbf{x}, y, f, t).$$

Let $p \in G$ such that $\mathsf{Dom}\, p$ contains all elements of $\mathbf{x}$. Since $D$ is dense, let $d \leq p$ with $d \in D$. Next, we'll need a $c \in G$ which has the same range as $d$. Take $c = j \restriction z$, where $z = j^{-1}[\mathsf{Ran}(d)]$. By definition of $\mathbb{P}$, $\mathsf{Ran}(d)$ is well-orderable in $N'$. Since $j : N \to N'$ is an isomorphism, $z$ is well-orderable in $N$. It follows that $c = j \restriction z$ is in $\mathbb{P}$ (and hence in $G$). Observe that $\mathsf{Ran}(c) = \mathsf{Ran}(d)$, and consequently $\tilde{c} = \tilde{d}$. Also, $c \leq p$, since both are in $G$ and $\mathsf{Ran}(c) = \mathsf{Ran}(d) \supseteq \mathsf{Ran}(p)$.

Since $d \in D$, we have $N \models \varphi(\mathbf{x}, y, f, d)$, and hence $N' \models \varphi(d(\mathbf{x}), y', f', \tilde{d})$ by $(*)$. But $\tilde{d} = \tilde{c}$, and $d(\mathbf{x}) = c(\mathbf{x})$ since both $d$ and $c$ extend $p$. $N' \models \varphi(c(\mathbf{x}), y', f', \tilde{c})$ and $N \models \varphi(\mathbf{x}, y, f, c)$. Therefore $c \in D \cap G$, which is what we needed to show that $G$ is generic. □

# 4 Homogeneity

Suppose in Theorem 1 that we insist in condition (a) that $N$ be a permutation submodel of $M$ in the more traditional sense, with $G \in M$. How must (b) and (c) be restricted to preserve equivalence? The answer for (b) was determined in [3]; the result is as follows.

**Theorem 6.** *Let $M$ be a transitive model of ZFAC, and let $N$ be a transitive subclass of $M$ which is a model of ZFA, such that $N$ and $M$ have the same set of atoms and the same pure part. The following are equivalent:*

(a′) *$N$ is a permutation submodel of $M$ obtained from a group $G \in M$.*

(b′) *$M$ is a generic extension of $N$ by an almost homogeneous notion of forcing.*

We now present a condition (c′) which is equivalent to (a′) and (b′), analogous to (c) in Theorem 1. This (c′) will say that $N$ has an SVC witness with a certain homogeneity property.

**Definition 1.** Working in ZFA, let $T$ be a set, and let $\mathbf{x}_1$, $\mathbf{x}_2$ be (transfinite) sequences of elements of $T$. We say that $\mathbf{x}_1$ and $\mathbf{x}_2$ have the same *$T$-type* if they satisfy the same $\Delta_1$ formulas using $T$ and pure sets as parameters. We say $\mathbf{x}_1$ and $\mathbf{x}_2$ are *$T$-isomorphic* if there is an $\in$-automorphism $F$ of $T$ such that $F(\mathbf{x}_1) = \mathbf{x}_2$.

**Theorem 7.** *Under the hypotheses of Theorem 6, conditions (a′) and (b′) are equivalent to*

(c′) *$N$ satisfies SVC with a set $S$ whose power set $T = (\mathscr{P}(S))^N$ has the following homogeneity property: Any two sequences in $N$ with the same $T$-type are $T$-isomorphic in $M$.*

**Example 1.** We consider the basic Fraenkel model and the ordered Mostowski model; see [5] for precise descriptions.

The basic Fraenkel model $N$ is the minimal model of ZFA for a given pure part and set of atoms. In $N$ the set $S$ of finite sequences of atoms is an SVC witness. (To see this, check that forcing with $S$ yields a generic extension satisfying AC, and use Theorem 2). Sequences in $N$ of elements of $T = \mathscr{P}(S)$ are finite, and so it is not hard to see that sequences in $N$ with the same $T$-type are $T$-isomorphic, not only in some $M$ where AC holds, but in $N$.

The above example is not typical. Suppose $N$ is the ordered Mostowski model, a minimal model of ZFA such that $A$ has a dense linear order $\prec$, obtained as a permutation submodel of some $M$ where $A$ is countable. The set $S$ of finite partial order embeddings $A \to \mathbb{Q}$ is an SVC witness. In $N$, there are no nontrivial automorphisms of $\langle A, \prec \rangle$. As a result, although sequences of elements of $S$ with the same $S$-type are $S$-isomorphic in $M$, they are not usually $S$-isomorphic in $N$ (and the same is true with $S$ replaced by $T = \mathscr{P}(S)$).

*Proof of Theorem 7.* Let $M$ and $N$ be as in the hypotheses of Theorem 6. First, assume that (c$'$) holds: $N$ satisfies SVC with $S$, and $T = (\mathscr{P}(S))^N$ satisfies the given homogeneity condition. We'll prove (b$'$). In $N$, define a notion of forcing $\mathbb{P}_1$ consisting of partial embeddings $T \to T'$, just as in the proof of Theorem 1, but replace $(*)$ with

$$N \models \phi(\mathbf{x}, p, y, f, T) \quad \leftrightarrow \quad N' \models \phi(p(\mathbf{x}), \tilde{p}, y', f', T'),$$

and further require that the above hold not only for $\Delta_0$ formulas, but rather all $\Delta_1$ formulas. The proof that $M$ is a $\mathbb{P}_1$-generic extension of $N$ works as before; it remains to show that $\mathbb{P}_1$ is an almost homogeneous notion of forcing. Observe that $T'$ is almost homogeneous in $M'$, and every $g \in \mathsf{Aut}(T', \in)$ induces a $\hat{g} \in \mathsf{Aut}(\mathbb{P}_1, \leq)$ by $(\hat{g}p)(x) = g(p(x))$.

Let $p$ and $q$ be conditions in $\mathbb{P}_1$, and let $\mathbf{d}$ and $\mathbf{e}$ be well-orderings of their respective domains. Following the proof of Theorem 1, we have a fixed isomorphism (the priming function) $j : M \to M'$ in $M$. Now $j(\mathbf{d})$ and $p(\mathbf{d})$ must have the same type in $T'$, so let $g$ be an automorphism of $T'$ such that $g(p(\mathbf{d})) = j(\mathbf{d})$. Likewise, find $h$ such that $h(q(\mathbf{e})) = j(\mathbf{e})$. Then $\hat{g}(p)$ maps $\mathbf{d}$ to $j(\mathbf{d})$, and $\hat{h}(q)$ maps $\mathbf{e}$ to $j(\mathbf{e})$. Thus $\hat{g}(p)$ and $\hat{h}(q)$ are compatible, which shows that $\mathbb{P}_1$ is almost homogeneous.

Conversely, assume that (a$'$) holds: $N$ is a permutation submodel of $M$, by a group $G \in M$. It is shown in [3] (Lemma 4.8) that $M$ is a generic extension of $N$ by a notion of forcing called the *generator poset*, which we'll denote by $\mathbb{P}_G$. The rest of the proof will make use of several facts about $\mathbb{P}_G$ proved in [3]; hereafter lemma and theorem numbers refer to that paper. There is a $\mathbb{P}_G$-name $\dot{f}$ which is $\Delta_0$-definable (from $\mathbb{P}_G$) such that $\mathbf{1}_{\mathbb{P}_G} \Vdash$ "$\dot{f} : \check{A} \to \check{A}'$ is a bijection" (Lemma 4.8), where $A'$ is a pure set as in Lemma 4; this $\dot{f}$ can be thought of as a name for an isomorphism $N \to N'$. SVC holds in $N$ with $\mathbb{P}_G$ (follows easily from Theorem 4.13), and thus SVC holds with $S = \mathbb{P}_G \cup \{\langle \mathbb{P}_G, \leq \rangle\}$. Let $T = (\mathscr{P}(S))^N$, and let $\mathbf{x}_1, \mathbf{x}_2 \in N$ be sequences of elements of $T$ such that $\mathbf{x}_1$ and $\mathbf{x}_2$ have the same $T$-type. We want to show that $\mathbf{x}_1$ and $\mathbf{x}_2$ are $T$-isomorphic.

Let $y$ be a pure set such that $p_1 \Vdash \dot{f}(\check{\mathbf{x}}_1) = \check{y}$ for some $p_1 \in \mathbb{P}_G$. By definition of "same $T$-type," it must also be true that there is a $p_2 \in \mathbb{P}_G$ such that $p_2 \Vdash \dot{f}(\check{\mathbf{x}}_2) = \check{y}$. By Lemma 4.10(c), there are filters $\Gamma_1$ and $\Gamma_2$ in $\mathbb{P}_G$, generic over $N$, such that $p_i \in \Gamma_i \in M$. Let $f_i = \mathsf{Val}_{\Gamma_i}(\dot{f})$; then $g = f_2^{-1} \circ f_1$ is in $G$ (by Lemma 4.10(d)). Since $G$ is a group of $\in$-automorphisms of $N$ and acts on $\langle \mathbb{P}_G, \leq \rangle$ (Lemma 4.6), we have that $g$ is an $\in$-automorphism of $T$, and observe that $g(\mathbf{x}_1) = \mathbf{x}_2$. $\square$

# References


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