

Unions and the Axiom of Choice

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Abstract

We study statements about countable and well ordered unions and how they are related to each other and to countable and well ordered forms of the axiom of choice.

1 Introduction and Definitions

In this paper we study the relationships between statements about countable and well ordered unions and the axiom of choice for families of countable and well ordered sets. All proofs, unless otherwise stated, are in ZF, Zermelo-Fraenkel set theory without AC (the axiom of choice).

In [14, p 158 ff], several theorems about unions are given which are equivalent to AC.

AC1 : $(\forall x)[x \prec \bigcup_{n \in \omega} A_n \rightarrow (\exists m \in \omega)(x \preceq \bigcup_{n \leq m} A_n)]$

AC2 : For any index set J , $(\forall j \in J, A_j \prec B_j) \rightarrow \bigcup_{j \in J} A_j \prec \prod_{j \in J} B_j$.

AC3 : For any index set J , $(\forall j \in J, 2 \preceq A_j) \rightarrow \bigcup_{j \in J} A_j \preceq \prod_{j \in J} A_j$.

The union theorems in this paper are weaker than AC.

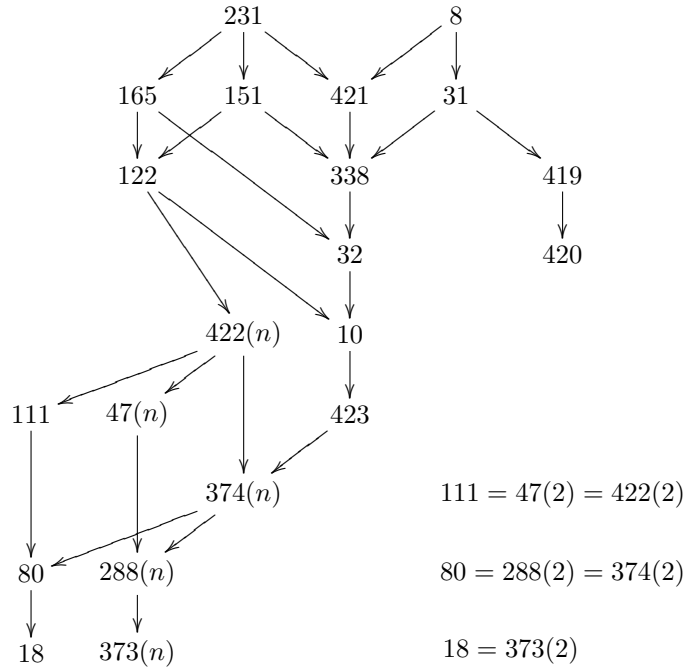
Some abbreviations will be used repeatedly for statements which are consequences of AC, as illustrated by the following examples. UT is a union theorem, PUT is a partial union theorem, C is a statement asserting the existence of choice functions, and PC is a partial choice theorem:

- UT($\aleph_0, < \aleph_0$, WO): The union of a countable number of finite sets can be well ordered.
- UT(\aleph_0, \aleph_0 , cuf): Every countable union of countable sets is *cuf*—that is, a countable union of finite sets.
- PUT($\aleph_0, \infty, \aleph_0$): The union of a countable number of infinite sets has a countably infinite subset.
- PC(WO, \aleph_0, ∞): Every well ordered family of countable sets has an infinite subfamily with a choice function.
- C(\aleph_0, ∞): Every countable set of infinite sets has a choice function.

The statements and equivalent forms of the statements that we will be studying are given below. The number of each statement is the number given in [6].

8. $\text{PUT}(\aleph_0, \infty, \aleph_0) \iff \text{PUT}(\text{WO}, \infty, \text{WO}) \iff \text{C}(\aleph_0, \infty) \iff \text{PC}(\infty, \infty, \infty) \iff \text{PC}(\aleph_0, \infty, \infty)$
10. $\text{UT}(\aleph_0, < \aleph_0, \aleph_0) \iff \text{UT}(\aleph_0, < \aleph_0, \text{WO}) \iff \text{PUT}(\text{WO}, < \aleph_0, \text{WO}) \iff \text{PUT}(\aleph_0, < \aleph_0, \aleph_0) \iff \text{C}(\aleph_0, < \aleph_0) \iff \text{PC}(\aleph_0, < \aleph_0, \infty) \iff \text{PC}(\text{WO}, < \aleph_0, \infty)$
18. $\text{PUT}(\aleph_0, 2, \aleph_0) \iff \text{PUT}(\text{WO}, 2, \text{WO}) \iff \text{PC}(\aleph_0, 2, \infty) \iff \text{PC}(\text{WO}, 2, \infty)$
31. $\text{UT}(\aleph_0, \aleph_0, \aleph_0)$
32. $\text{C}(\aleph_0, \aleph_0) \iff \text{PC}(\aleph_0, \aleph_0, \infty) \iff \text{PC}(\text{WO}, \aleph_0, \infty)$
- 47(n) [$n \geq 2$]. $\text{C}(\text{WO}, n)$
80. $\text{UT}(\aleph_0, 2, \aleph_0) \iff \text{UT}(\aleph_0, 2, \text{WO}) \iff \text{C}(\aleph_0, 2)$
111. $\text{UT}(\text{WO}, 2, \text{WO}) \iff \text{C}(\text{WO}, 2)$
122. $\text{UT}(\text{WO}, < \aleph_0, \text{WO}) \iff \text{C}(\text{WO}, < \aleph_0)$
151. $\text{UT}(\text{WO}, \aleph_0, \text{WO})$
165. $\text{C}(\text{WO}, \text{WO})$
231. $\text{UT}(\text{WO}, \text{WO}, \text{WO})$
- 288(n) [$n \geq 2$]. $\text{C}(\aleph_0, n)$
338. $\text{UT}(\aleph_0, \aleph_0, \text{WO})$
- 373(n) [$n \geq 2$]. $\text{PUT}(\aleph_0, n, \aleph_0) \iff \text{PUT}(\text{WO}, n, \text{WO}) \iff \text{PUT}(\aleph_0, n, \text{WO}) \iff \text{PC}(\text{WO}, n, \infty) \iff \text{PC}(\aleph_0, n, \infty)$
- 374(n) [$n \geq 2$]. $\text{UT}(\aleph_0, n, \aleph_0) \iff \text{UT}(\aleph_0, n, \text{WO}) \iff (\forall i, 2 \leq i \leq n) \text{C}(\aleph_0, i)$
419. $\text{UT}(\aleph_0, \text{cuf}, \text{cuf}) \iff \text{UT}(\text{cuf}, \text{cuf}, \text{cuf})$
420. $\text{UT}(\aleph_0, \aleph_0, \text{cuf})$
421. $\text{UT}(\aleph_0, \text{WO}, \text{WO})$
- 422(n) [$n \geq 2$]. $\text{UT}(\text{WO}, n, \text{WO}) \iff (\forall i, 2 \leq i \leq n) \text{C}(\text{WO}, i)$
423. $(\forall n \geq 2) \text{C}(\aleph_0, n) \iff (\forall n \geq 2) 374(n) \iff (\forall n \geq 2) 288(n)$

Known implications between the forms are given in the diagram below:



In this diagram, “ $422(n) \rightarrow 111$ ” means “ $(\forall n)[422(n) \rightarrow 111]$ ”; “ $422(n) \rightarrow 374(n)$ ” means “ $(\forall n)[422(n) \rightarrow 374(n)]$ ”, and so on (where n ranges over $\omega \setminus \{0, 1\}$). Forms 47(n) and 111 and all forms above those are statements about well-orderable families of sets; all other forms are statements about countable families of sets.

It is clear that form 31, $\text{UT}(\aleph_0, \aleph_0, \aleph_0)$, implies that cuf sets are countable, so 31 implies form 419, $\text{UT}(\aleph_0, \text{cuf}, \text{cuf})$. Also, form 8, $\text{C}(\aleph_0, \infty)$, implies 421, $\text{UT}(\aleph_0, \text{WO}, \text{WO})$, because given a countable family of well orderable sets, form 8 can be used to choose a well ordering on each of the sets in the family. The proof that 8 implies 31 is similar. All the other implications in the diagram above are clear.

Most independence proofs will make use of permutation models (Fraenkel-Mostowski models), which are models of the theory ZFA (ZF modified to allow atoms). Alternately, permutation models may be formulated to be models of ZF^0 , which is ZF minus the axiom of regularity. See Jech [7] for basics about permutation models and definitions of relevant terminology such as *support* and *symmetric* in the context of permutation models (see also the proof of Theorem 2.2 for examples). If G acts on a set X and $x \in X$, we write G_x for the stabilizer of x and $G_{(x)}$ for the pointwise stabilizer of x .

2 Well-orderable Families of Well-orderable Sets

In this section, we consider some consequences of $\text{UT}(\text{WO}, \text{WO}, \text{WO})$ (form 231). The main new results are that $\text{UT}(\text{WO}, \aleph_0, \text{WO})$ (form 151) does not imply $\text{UT}(\aleph_0, \text{WO}, \text{WO})$ (form 421), and that

$C(\text{WO}, \text{WO})$ (form 165) does not imply $\text{UT}(\aleph_0, \aleph_0, \text{WO})$ (form 338). Both of these results are obtained with models of ZFA, and some of the models in this section are not Fraenkel-Mostowski models in the usual sense. It may therefore not be straightforward to transfer these results to ZF.

Some known theorems about subgroups of “small index” in certain permutation groups will be of help in deducing properties the models in this section. For example:

Theorem 2.1 (Gaughan [4]). *Every proper subgroup of the symmetric group $\text{Sym}(\aleph_1)$ has uncountable index.*

Theorem 2.2. *There is a model of ZF^0 in which $\text{UT}(\text{WO}, \aleph_0, \text{WO})$ (form 151) is true, while $C(\aleph_0, \text{WO})$ is false (and hence forms 421 and 165 are false).*

Proof. Toward construction of a permutation model, start with a model \mathcal{M} of ZFAC in which $\{A_i : i \in \omega\}$ is a partition of the set A of atoms into sets of size \aleph_1 . For each $i \in \omega$, let $G_i = \text{Sym}(A_i)$; we consider these to be subgroups of $\text{Sym}(A)$ in the natural way. Let G be the direct sum $\bigoplus_i G_i$ (so each $g \in G$ will fix pointwise all but finitely many of the A_i 's). As supports, take sets of the form $A_0 \cup \dots \cup A_n$ (and, as usual, we say that E supports $x \in \mathcal{M}$ if $g(x) = x$ whenever g is in the pointwise stabilizer of E). Let \mathcal{N} be the resulting permutation model (i.e., \mathcal{N} is the class of hereditarily symmetric elements of \mathcal{M} , where $x \in \mathcal{M}$ is *symmetric* if x has a support of the form just described.)

In \mathcal{N} , it is clear that $\{A_i : i \in \omega\}$ is a countable set of well-orderable sets with no choice function, and thus $C(\aleph_0, \text{WO})$ is false.

Next, we show that if $E = A_0 \cup \dots \cup A_n$ is a support for a countable set $C \in \mathcal{N}$, then in fact E supports each element of C . Let $x \in C$, and let $m \in \omega \setminus n$. For $\pi \in G_m$, we have $\pi C = C$, so the G_m -orbit of x is contained in C and is therefore countable. By Theorem 2.1, the size of the G_m -orbit of x (equal to the index of the stabilizer of x in G_m) must be 1. Since $G_{(E)} = \bigoplus_{n < m < \omega} G_m$, it follows that the $G_{(E)}$ -orbit of x has size 1; in other words, E supports x .

Now we can see that $\text{UT}(\text{WO}, \aleph_0, \text{WO})$ holds in \mathcal{N} . For if $\{C_\alpha : \alpha \in \kappa\}$ is some well-ordered family of countable sets in \mathcal{N} , then there is an E that is a support for each C_α . By the previous paragraph, this E supports each element of each C_α , and is therefore a support for a well-ordering of $\bigcup_{\alpha \in \kappa} C_\alpha$. \square

Next, we want to find a model of $C(\text{WO}, \text{WO}) + \neg \text{UT}(\aleph_0, \text{WO}, \text{WO})$. In the model of the previous proof, $C(\text{WO}, \text{WO})$ is false since there is no choice function for the A_i 's. It may be tempting to try to modify that model by adding some choice functions as supports. However, the next theorem shows that no traditional permutation model can have the desired properties.

Theorem 2.3. *In models of ZFA with AC holding in the pure part (or models of ZF^0 with AC holding in the well-founded part): Form 165, $C(\text{WO}, \text{WO})$ implies form 231, $\text{UT}(\text{WO}, \text{WO}, \text{WO})$ (and hence these forms are equivalent in such models).*

Proof. AC restricted to pure sets is equivalent to “the power set of a well-orderable set is well-orderable” (form 91 in [6]). Working in ZFA, assume form 91 + form 165.

Let $\langle A_\alpha \rangle_{\alpha \in \kappa}$ be a family of well-orderable sets indexed by ordinals. By form 91, $\wp(A_\alpha)$ is well-orderable, and hence so is the set W_α of well-order relations of A_α , since W_α can be mapped

injectively to $\wp(A_\alpha)$. By $C(\text{WO}, \text{WO})$, there is a sequence of well-orderings $\langle w_\alpha \rangle_{\alpha \in \kappa}$ where $w_\alpha \in W_\alpha$ for each α ; from this sequence a well-ordering of $\bigcup_\alpha A_\alpha$ may easily be obtained. \square

In view of Theorem 2.3, our solution to the problem of finding a model of $C(\text{WO}, \text{WO}) + \neg\text{UT}(\aleph_0, \text{WO}, \text{WO})$ will be to not insist that AC hold in the pure part of our permutation model. We can construct a permutation model in the usual way, except that we start with a model of ZFA whose pure part is a model of $\text{ZF} + \neg\text{AC}$, carefully chosen.

Howard [5] has described a permutation model, called $\mathcal{N}18$ in [6], in which $C(\text{WO}, \aleph_0)$ is true, but $\text{UT}(\aleph_0, \aleph_0, \aleph_0)$ is false. By Theorem 2.3, $C(\text{WO}, \text{WO})$ is also false in $\mathcal{N}18$. However, a modification of $\mathcal{N}18$ in the spirit of the previous paragraph will yield a model of $C(\text{WO}, \text{WO}) + \neg\text{UT}(\aleph_0, \text{WO}, \text{WO})$. Start with a model \mathcal{M} of ZFA with the following four properties: (i) $\mathcal{M} \models C(\text{WO}, \text{WO})$, (ii) there is an ultrafilter on ω , (iii) 2^{\aleph_0} is not well-orderable, and (iv) A is countable. For example, one can obtain such a model by adjoining a countable set of atoms to the basic Cohen model. (The basic Cohen model is described in [7], or see $\mathcal{M}1$ in [6].) Now define a permutation submodel of \mathcal{M} following the description of $\mathcal{N}18$ in [5] (which relies on an ultrafilter on ω). Using the fact that $C(\text{WO}, \text{WO})$ holds in the pure part and the fact that 2^{\aleph_0} is not well-orderable, the argument in [5] that $C(\text{WO}, \aleph_0)$ holds in $\mathcal{N}18$ can be modified fairly easily to show that $C(\text{WO}, \text{WO})$ holds in our modified version of $\mathcal{N}18$.

Instead of relying on a modification of the proof in [5], we will present a complete proof of the independence result, based on a new (but similar) model. We will later modify the new proof to obtain further results.

The new proof will rely on this group theoretic result:

Theorem 2.4 (Truss, [13]). *Let $G = \text{Aut}(\mathbb{Q})$, the group of autohomeomorphisms of \mathbb{Q} . If $H \leq G$ such that $[G : H]$ is well-orderable and $[G : H] \not\leq 2^{\aleph_0}$, then there is a finite $\Delta \subset \mathbb{Q}$ such that $G_{(\Delta)} \leq H$, where $G_{(\Delta)}$ is the pointwise stabilizer of Δ .*

Remarks on the proof. Truss implicitly assumes AC in the paper [13], from which this result is taken. The original version of this theorem, stated as Theorem 2.12 of [13], does not say anything about $[G : H]$ being well-orderable, and the hypothesis reads “ $[G : H] < 2^{\aleph_0}$ ” instead of “ $[G : H] \not\leq 2^{\aleph_0}$.” (Indeed, the concern of the paper is subgroups of “small” index in permutation groups, i.e. index strictly less than 2^{\aleph_0} .) Nevertheless, the proof in [13] gives the generalization stated here, in ZF. The well-orderability of \mathbb{Q} and the fact that \mathbb{Q} has a well-orderable base of clopen sets allows one to avoid AC when apparently arbitrary choices are called for in the proof.

We make one specific comment on how well-orderability of $[G : H]$ comes up in the proof. Lemma 2.2 of [13] contains an argument of the following form: Given $\prod_{i \in \omega} |S_i| < 2^{\aleph_0}$, it is concluded that at all but finitely many of the S_i must have cardinality ≤ 1 . (We are using S_i as an abbreviation for “ $[K_i : H_i]$ ” in Lemma 2.2 of [13]. It would be an unnecessary diversion for us to discuss what K_i and H_i actually refer to in [13].) The conclusion is valid assuming AC, but without AC one might worry that the product on the left hand side is small just due to a lack of choice functions, even if the S_i are all large. However, in the particular context of the Lemma in [13], we actually have $1 \leq \prod |S_i| \leq [G : H]$, and $[G : H]$ is assumed to be well-orderable. It follows that there is a uniform well-ordering of all the S_i 's, and hence the conclusion that all but finitely many S_i 's must

be singletons does not require AC after all. (Also note that the conclusion still holds if “ $< 2^{\aleph_0}$ ” is replaced by “ $\not\leq 2^{\aleph_0}$.”) \square

Corollary 2.5. *The statement of Theorem 2.4 still holds if $\text{Aut}(\mathbb{Q})$ is replaced everywhere with $\text{Aut}(\mathbb{Q})_0$, where $\text{Aut}(\mathbb{Q})_0$ is the stabilizer subgroup of 0 in $\text{Aut}(\mathbb{Q})$.*

Proof. Follows from the fact that $[\text{Aut}(\mathbb{Q}) : \text{Aut}(\mathbb{Q})_0] = \aleph_0$. \square

The following, more straightforward facts about $\text{Aut}(\mathbb{Q})$ are also useful. We will not prove them here. Lemma 2.6 of [13] is a stronger version of part (ii).

Lemma 2.6. *Let A, B be disjoint clopen subsets of \mathbb{Q} .*

- (i) *There is an autohomeomorphism of \mathbb{Q} which swaps A and B .*
- (ii) *If there is an interval $I \subseteq \mathbb{Q}$ disjoint from $A \cup B$, then the group $\text{Aut}(\mathbb{Q})$ is generated by the pointwise stabilizer of A together with the pointwise stabilizer of B .*

Theorem 2.7. *There is a model of ZFA in which form 165, $C(\text{WO}, \text{WO})$, is true, while form 338, $UT(\aleph_0, \aleph_0, \text{WO})$, is false.*

Proof. Let \mathcal{M} be a model of ZFA in which the set A of atoms is countable, and in which the pure part is a copy of the basic Cohen model (model $\mathcal{M}1$ in [6]). Thus, in \mathcal{M} , $C(\text{WO}, \text{WO})$ holds, and 2^{\aleph_0} is not well-orderable. Let $\{A_i : i \in \omega\}$ be a partition of A into countably many countable sets, and for each $i \in \omega$, fix an element $0_i \in A_i$ and a topology τ_i on A_i so that A_i is homeomorphic to \mathbb{Q} . Let $G = \prod_{i \in \omega} \text{Aut}(A_i, \tau_i, 0_i)$; that is, G is the group of all permutations of A which act on each A_i by autohomeomorphisms fixing 0_i . Let supports be sets of atoms of the form $E = \bigcup_{i \in \omega} E_i$ such that $E_i = A_i$ for at most finitely many i , and such that for the rest of the $i \in \omega$, $E_i \subset A_i$ is a complement of a neighborhood of 0_i . Let \mathcal{N} be the permutation submodel of \mathcal{M} obtained by this group and collection of supports. It is clear that form 338 is false in \mathcal{N} , since $\{A_i : i \in \omega\}$ is a countable set of countable sets whose union A is not well-orderable.

It remains to show that $\mathcal{N} \models C(\text{WO}, \text{WO})$, so let $\{W_\alpha : \alpha \in \kappa\} \in \mathcal{N}$ be a well-ordered family of sets, each of which is well-orderable in \mathcal{N} ; our goal is to find a choice function for this family. Let E be a support for the family, so that in fact E supports each W_α . Let $E_i = E \cap A_i$, and fix $k \in \omega$ such that $E_i \neq A_i$ for all $i \geq k$. By enlarging E if necessary, assume $E_i = A_i$ whenever $i < k$, and assume E_i is clopen in A_i for each $i \in \omega$.

For all $i \geq k$, let $B_i \subset A_i$ be a clopen set disjoint from E_i and disjoint from some neighborhood of 0_i , and let $B = \bigcup_i B_i$. To show that $\{W_\alpha : \alpha \in \kappa\}$ has a choice function in \mathcal{N} , we will show that for each $\alpha \in \kappa$ there is a $w \in W_\alpha$ such that $E \cup B$ supports w . Here’s why that will suffice: Using $C(\text{WO}, \text{WO})$ in \mathcal{M} , obtain a choice function f for the set $\{w \in W_\alpha : E \cup B \text{ supports } w\}$. Such an f is also a choice function for the W_α ’s, and is supported by $E \cup B$, so we have $f \in \mathcal{N}$, as desired.

Now, fix some $\alpha \in \kappa$. The existence of some $w \in W_\alpha$ supported by $E \cup B$, will be immediate from Claim 1 and Claim 3 below. Claim 2 is used to establish Claim 3.

Claim 1: If $x \in W_\alpha$ and there is a support S for x such that $S \cap A_i \neq A_i$ whenever $i \geq k$, then there exists $w \in W_\alpha$ supported by $E \cup B$.

Proof of claim. For each $i \geq k$, let $B'_i = S \cap A_i$ and let $B = \bigcup_i B'_i$. Let $\sigma \in G$ such that σ fixes E pointwise and such that $\sigma B' \subseteq B$. (The existence of such a σ follows from Lemma 2.6(i) and the fact that each E_i is clopen and each B_i contains an interval.) Let $w = \sigma x$. Since $\sigma \in G_{(E)}$, we have $\sigma W_\alpha = W_\alpha$, so $w \in W_\alpha$. And since S supports x , σS supports w . But $\sigma S \subseteq E \cup B$.

Claim 2: There is a $j \geq k$ such that $E \cup A_k \cup \dots \cup A_j$ supports every element of W_α .

Proof of claim. Since W_α is well-orderable, there is a support D which supports every element of W_α . We may assume that $D \supseteq E$ and that $D_i := D \cap A_i$ is clopen for each $i \in \omega$. Let $j \in \omega$ such that D_i is a proper subset of A_i for each $i > j$. Let $\sigma \in G_{(E)}$ such that for each $i > j$, $(D_i \setminus E_i) \cap \sigma(D_i \setminus E_i) = \emptyset$. Since $\sigma W_\alpha = W_\alpha$, σD is another support for all the elements of W_α . By repeated applications of Lemma 2.6(ii), we find that every element of G which fixes $E \cup A_k \cup \dots \cup A_j$ pointwise can be written as a product of elements of $G_{(D)} \cup G_{(\sigma D)}$, and these group elements do not move any elements of W_α .

Claim 3: If $x \in W_\alpha$, then there is a support S for x such that $S \setminus E$ is finite.

Proof of claim. Let H_i be the set of elements of G which only move elements of $A_i \setminus E$; note that $H_i \cong \text{Aut}(\mathbb{Q})_0$. Let $H = \prod_{k \leq i \leq j} H_i$, where $j \geq k$ is as in Claim 2. Since $H \subset G_{(E)}$, we have $\eta x \in W_\alpha$ for all $\eta \in H$. Since W_α is well-orderable, so is the H_i -orbit of x for each i , and hence so is the index of the stabilizer $(H_i)_x$ in H_i . Since 2^{\aleph_0} is not well-orderable, we have $[H_i : (H_i)_x] \not\leq 2^{\aleph_0}$. By Corollary 2.5 there is a finite $S_i \subset A_i \setminus E_i$ such that the pointwise stabilizer $H_{i(S_i)}$ is contained in $(H_i)_x$. In other words, S_i supports x in the group H_i . Let $S' = \bigcup_{k \leq i \leq j} S_i$. Since $G_{(E \cup S')} = H_{k(S_k)} \times \dots \times H_{j(S_j)} \times G_{(E \cup A_0 \cup \dots \cup A_j)}$, and this is a finite product of groups which fix x (by Claim 2), we have that $S := E \cup S'$ is a support for x . This ends the proof of Claim 3, and of the theorem. □

The proof of Theorem 2.7 gave a model in which $C(\text{WO}, \text{WO})$ is true, while $\text{UT}(\aleph_0, \text{WO}, \text{WO})$ is false, because $\text{UT}(\aleph_0, \aleph_0, \text{WO})$ is false. We next present a model of $C(\text{WO}, \text{WO}) + \neg \text{UT}(\aleph_0, \text{WO}, \text{WO})$ in which $\text{UT}(\aleph_0, \aleph_0, \text{WO})$ still holds—in fact, $\text{UT}(\aleph_0, \aleph_0, \aleph_0)$ will hold. This will require a permutation submodel based on a group acting on uncountable well-orderable sets of atoms, and so Truss' result (Theorem 2.4) does not seem to be of use. The following group theoretic result will be used instead. We state the theorem in terms of a general well-ordered cardinal \aleph , but we'll only directly make use of it in the case $\aleph = \aleph_1$.

Theorem 2.8 (Dixon, Neumann, Thomas [3]). *Let \aleph be an aleph, and let G be a subgroup of $S := \text{Sym}(\aleph)$ with $[S : G]$ well-orderable and $[S : G] \not\leq 2^\aleph$.*

- (i) *If there is an almost disjoint family $F \subset \wp(\aleph)$ such that $[S : G] \not\leq |F|$, then there is a $\Delta \subset \aleph$ with $|\Delta| < \aleph$ such that $S_{(\Delta)} \leq G$.*
- (ii) *If $2^{<\aleph} = \aleph$, then there is a $\Delta \subset \aleph$ with $|\Delta| < \aleph$ such that $S_{(\Delta)} \leq G$.*

Remarks on the proof. For part (i), see the proof of Theorem 2 (and Theorem 2b) of [3], a paper which explicitly assumes AC (and sometimes GCH). The proof there yields the statements given here in ZF. The Lemma in Section 2 of [3] is similar to (and perhaps the model for) Lemma 2.6 of

[13], which we commented on following the statement of our Theorem 2.4. The remarks we gave there apply also to the Lemma in [3].

Part (ii) is just a corollary of part (i), since the exponentiation hypothesis of part (ii) yields an almost disjoint family as required by the hypothesis of part (i) (see, e.g. [8] Ch 9). \square

Corollary 2.9. *In any model of ZF+CH in which 2^{\aleph_1} is not well-orderable, whenever H is a subgroup of $S := \text{Sym}(\aleph_1)$ with well-orderable index, there is a countable $\Delta \subset \aleph_1$ such that $S_{(\Delta)} \leq H$.*

Lemma 2.10. *There is a model of ZF+CH+C(WO,WO) in which 2^{\aleph_1} is not well-orderable. (CH means $2^{\aleph_0} = \aleph_1$.)*

Proof. We define a symmetric model which is a variation on the basic Cohen model as described in [7], where instead of adding Cohen reals to a ground model, we shall add new subsets of ω_1 .

Start with a model \mathcal{N} of ZFC + CH as the ground model. Let \mathbb{P} be the notion of forcing consisting of partial functions $p : \omega_1 \times \omega \rightarrow 2$ such that $\text{Dom}(p) \subseteq \alpha \times n$ for some $\alpha < \omega_1$ and $n < \omega$. Thus forcing with \mathbb{P} adds an ω -indexed family of subsets of ω_1 . Let Γ be a filter in \mathbb{P} , generic over \mathcal{N} , and let \mathcal{M} be the symmetric submodel of $\mathcal{N}[\Gamma]$ based on the group G of all automorphisms of \mathbb{P} induced by permutations of ω , with finite supports. For more detail, see [7] and make the appropriate changes to the description of the basic Cohen model. The argument that \mathcal{M} has a non-well-orderable set of subsets of \aleph_1 is analogous to the argument that the basic Cohen model has a non-well-orderable set of reals, and the argument in [7] that $C(\infty, \text{WO})$ holds in the basic Cohen model also carries over to our modified construction.

To see that CH holds in \mathcal{M} , define $\mathbb{P}_m := \{p \in \mathbb{P} : \text{Dom}(p) \subset \omega_1 \times m\}$ for each $m \in \omega$. Then $\Gamma_m := \Gamma \cap \mathbb{P}_m$ is generic in \mathbb{P}_m . Observe that every new set of ordinals in \mathcal{M} lies in $\mathcal{N}[\Gamma_m]$ for some $m \in \omega$. Since forcing with \mathbb{P}_m adds no new reals, we see that the reals of \mathcal{M} are all in the ground model \mathcal{N} , which we chose to be a model of CH. \square

Theorem 2.11. *There is a model of ZFA in which forms 165 and 31 ($C(\text{WO}, \text{WO}) + \text{UT}(\aleph_0, \aleph_0, \aleph_0)$) are true, while form 421, $\text{UT}(\aleph_0, \text{WO}, \text{WO})$, is false.*

Proof. Start with a model \mathcal{M} of ZFA in which $|A| = \aleph_1$, and in which the pure part is a model of ZF+CH+C(WO,WO) in which 2^{\aleph_1} is not well-orderable, as given by Lemma 2.10. We construct a permutation submodel similar to the one in the proof of Theorem 2.7. Let $\{A_i : i \in \omega\}$ be a partition of A into countably many sets each of cardinality \aleph_1 . Let $G = \prod_{i \in \omega} \text{Sym}(A_i)$; i.e. G is the group of all permutations of A which fix each set A_i . Let supports be sets of atoms of the form $E = \bigcup_{i \in \omega} E_i$ such that $E_i = A_i$ for at most finitely many i , and such that for the rest of the $i \in \omega$, E_i is a countable subset of A_i . Let \mathcal{N} be the permutation submodel of \mathcal{M} obtained by this group and collection of supports. Clearly, in \mathcal{N} , A is a countable union of well-orderable sets which is not itself well-orderable, so $\text{UT}(\aleph_0, \text{WO}, \text{WO})$ is false.

We want to show that $\text{UT}(\aleph_0, \aleph_0, \aleph_0)$ and $C(\text{WO}, \text{WO})$ hold in \mathcal{N} . Observe that $\text{UT}(\aleph_0, \aleph_0, \aleph_0)$ follows (by the argument of Theorem 2.3) from $C(\text{WO}, \text{WO}) + "2^{\aleph_0} \text{ is well-orderable.}"$ Since $\mathcal{N} \models 2^{\aleph_0} = \aleph_1$ by our initial choice of pure part, it now remains to show that $\mathcal{N} \models C(\text{WO}, \text{WO})$.

Let $\{W_\alpha : \alpha \in \kappa\} \in \mathcal{N}$ be a well-ordered family of well-orderable sets, with a support $E = A_0 \cup \dots \cup A_{k-1} \cup (\bigcup_{i \geq k} E_i)$, where $E_i \subset A_i$ is countable whenever $i \geq k$. Let $B \subset A$ such that $B \cap A_i$ is countably infinite and disjoint from E_i for all $i \geq k$. Just as in the proof of Theorem 2.7, we can show that $E \cup B$ supports a choice function for the W_α 's. The proof is very closely analogous to that of Theorem 2.7, and we leave it to the interested reader to make the needed changes. Note that the analogue of Claim 3 would read, "If $x \in W_\alpha$, then there is a support S for x such that $S \setminus E$ is countable," and the proof in this situation would involve an invocation of Corollary 2.9 instead of Corollary 2.5. \square

Theorem 2.11 says that $C(\text{WO}, \text{WO}) + \text{UT}(\aleph_0, < \kappa, \text{WO})$ does not imply $\text{UT}(\aleph_0, \kappa, \text{WO})$ when $\kappa = \aleph_1$. The argument could be modified to give the same result for higher κ , using Theorem 2.8 (the group theoretic result of Dixon et. al.) with $\aleph = \kappa$. However, although Theorem 2.8 is interesting in the case $\aleph = \aleph_0$, there does not seem to be a way to use that case to get a model of $C(\text{WO}, \text{WO}) + \neg \text{UT}(\aleph_0, \aleph_0, \text{WO})$, as in our Theorem 2.7.

3 Cuf Sets

A set is *cuf* if it is a countable union of finite sets. It was observed in [2] that $\text{UT}(\aleph_0, \text{cuf}, \text{cuf})$ and $\text{UT}(\aleph_0, \aleph_0, \text{cuf})$ (forms 419 and 420) are strictly intermediate in strength between form 418 (a countable disjoint union of metrizable spaces is metrizable) and form 34 (\aleph_1 is regular). It was left open in that paper whether forms 419 and 420 are equivalent; we will show in Theorem 3.3 that they are not.

It was observed in [2] that forms 419 and 420 hold in the second Fraenkel model (model $\mathcal{N}2$ of [6]). Here is a more thorough analysis.

Theorem 3.1. *In ZF^0 , form 67 (The axiom of multiple choice) implies form 419.*

Proof. We use the following version of form 67 (see [6]):

Form 67B: Every set may be covered by a well-ordered union of finite sets.

Assume 67B and let $X = \bigcup_{i \in \omega} X_i$, where each X_i is cuf. By 67B, let $\{F_\alpha : \alpha \in \beta\}$ be a partition of X into finite sets, where β is an ordinal. We want to show that β is countable.

Let $B_i = \{\alpha \in \beta : X_i \cap F_\alpha \neq \emptyset\}$. Since X_i is cuf, B_i is also cuf; since B_i is cuf and well-orderable, it is countable. Thus $\beta = \bigcup_i B_i$ is a countable union of countable sets. Since form 67 is known to imply that \aleph_1 is regular (form 34), β must be a countable ordinal. \square

Corollary 3.2. *Form 419 does not imply any of the forms listed in Section 1, except for itself and form 420.*

Proof. By the diagram in Section 1, it suffices to show that form 419 does not imply form 373(n) whenever $2 \leq n < \omega$. For each n , the permutation model $\mathcal{N}2(n)$ in [6] is a model of $67 + \neg 373(n)$, and therefore is a model of $419 + \neg 373(n)$ by Theorem 3.1. \square

In ZF, form 67 is equivalent to AC, so Theorem 3.1 is only useful in fragments such as ZFA or ZF⁰. However, the independence result “419 $\not\rightarrow$ 373(n)” in Corollary 3.2 is transferable to ZF using the results of Jech and Sochor [9, 10] and Pincus [11, 12] (even though the independence of 373(n) from 67 is not transferable). The following independence result is based on a permutation model, but it, too, is transferable to ZF by these methods.

Theorem 3.3. *Form 420, UT($\aleph_0, \aleph_0, \text{cuf}$), does not imply form 419, UT($\aleph_0, \text{cuf}, \text{cuf}$).*

Proof. Let \mathcal{N} be the Fraenkel-Mostowski model determined by the set A of atoms, the group G of permutations and the filter \mathfrak{F} of subgroups of G described below. Let A have a partition $\{A_i : i \in \omega\}$ into countably many infinite sets, and for each $i \in \omega$, let A_i have a partition $P_i = \{A_{i,j} : j \in \omega \setminus \{0\}\}$ into countably many finite sets with $|A_{i,j}| = j$. Let G_i be the group of all finite permutations of A_i ; we identify G_i with the group of all permutations of A which only move only finitely many elements of A_i and fix all other atoms. Let G be the direct sum of the G_i 's, so that

$$G = \{ \phi \in \text{Sym}(A) : \forall i \in \omega, \phi(A_i) = A_i \text{ and } \{a \in A : \phi(a) \neq a\} \text{ is finite} \}.$$

Let $\mathbf{P}_i = \{ \phi(P_i) : \phi \in G \}$, and let $\mathbf{P} = \bigcup_{i \in \omega} \mathbf{P}_i$. We define \mathfrak{F} to be the filter of subgroups of G generated by the subgroups $G_{(E)} = \{ \phi : \forall P \in E, \phi(P) = P \}$, where E ranges over the finite subsets of \mathbf{P} . We say that an element $x \in \mathcal{N}$ has support E if $(\forall \phi \in G_{(E)})(\phi(x) = x)$ and we refer to finite subsets of \mathbf{P} as *supports*. Note that for each $i \in \omega$, every element Q of \mathbf{P}_i is a partition of A_i into sets of different cardinalities. Therefore for any $\phi \in G$, ϕ fixes Q if and only if ϕ fixes Q pointwise. In addition, since each $\phi \in G$ moves only finitely many elements,

$$(\forall Q \in \mathbf{P}_i)(\exists j_Q \in \omega) Q = D \cup \{A_{i,j} : j > j_Q\}, \quad (*)$$

where D is a finite set disjoint from $\{A_{i,j} : j > j_Q\}$ and $\bigcup D = \bigcup \{A_{i,j} : j \leq j_Q\}$. (If $Q = \psi(P_i)$ then $D = \{ \psi(A_{i,j}) : (\exists a \in A_{i,j})(\psi(a) \neq a) \}$.)

To see that UT($\aleph_0, \text{cuf}, \text{cuf}$) is false in \mathcal{N} , observe that each A_i is cuf in \mathcal{N} (since $E = \{P_i\}$ is a support for an enumeration of P_i), and the enumeration $i \mapsto A_i$ is in \mathcal{N} (with empty support), so that A is a countable union of cuf sets.

On the other hand, A is not a countable union of finite sets in \mathcal{N} . The argument is by contradiction. Suppose that $A = \bigcup_{j \in \omega} B_j$ where each B_j is finite, and the enumeration $j \mapsto B_j$ is in \mathcal{N} with support E . This means that the $G_{(E)}$ -orbit $\{ \phi(a) : \phi \in G_{(E)} \}$ of each atom $a \in A$ is contained in some finite B_j . But since E is a finite subset of \mathbf{P} , there is some $i_0 \in \omega$ such that $\bigcup \bigcup E \cap A_{i_0} = \emptyset$, which implies that $G_{i_0} \subseteq G_{(E)}$. This is a contradiction, since the G_{i_0} -orbit of any atom $a \in A_{i_0}$ is the infinite set A_{i_0} .

Now we argue that UT($\aleph_0, \aleph_0, \text{cuf}$) is true in \mathcal{N} . The first step is to prove

Lemma. *For any Y in \mathcal{N} , if E is a support of Y and Y is well orderable in \mathcal{N} , then for all $y \in Y$, $\{ \phi(y) : y \in G_{(E)} \}$ is finite.*

Proof. Assume that $Y \in \mathcal{N}$ and that E is a support of Y . Also assume that $y \in Y$, Y is well orderable in \mathcal{N} , and that E_0 is a support of a well ordering of Y . This means that E_0 supports

every element of Y , and in particular E_0 supports y . As argued above, there is a $k \in \omega$ such that $\bigcup \bigcup E \subseteq \bigcup_{i=0}^k A_i$. Let $I = \{i \in \omega : E \cap \mathbf{P}_i \neq \emptyset\} = \{i \in \omega : \bigcup \bigcup E \cap A_i \neq \emptyset\}$; this is finite.

Sublemma. *For all $\phi \in G$, if ϕ fixes $\bigcup_{i \in I} A_i$ pointwise, then $\phi(y) = y$.*

Proof. The subgroup of permutations in G which fix $\bigcup_{i \in I} A_i$ pointwise is generated by $\bigcup_{i \notin I} G_i$, and it will suffice to show that permutations in that generating set fix y . To that end, fix $k' \in \omega \setminus I$ and assume $\phi \in G_{k'}$. Assume by way of contradiction that $\phi(y) \neq y$.

Now we consider $E_0 \cap \mathbf{P}_{k'}$. It follows from (*) that there is a $j_0 \in \omega$ such that for every $Q \in E_0 \cap \mathbf{P}_{k'}$, $Q = D_Q \cup \{A_{k',j} : j > j_0\}$ where for each $Q \in E_0 \cap \mathbf{P}_{k'}$, D_Q is finite and disjoint from $\{A_{k',j} : j > j_0\}$. We choose such a j_0 which we also assume is chosen large enough so that ϕ fixes $\bigcup_{j > j_0} A_{k',j}$ pointwise.

Now choose $n \in \omega$, such that $n > \left| \bigcup_{j=1}^{j_0} A_{k',j} \right|$. Since $|A_{k',j_0}| = j_0$ we have $n > j_0$ and therefore that $A_{k',n} \in Q$ for every $Q \in E_0 \cap \mathbf{P}_{k'}$. Let f be a one-to-one function from $\bigcup_{j=1}^{j_0} A_{k',j}$ to $A_{k',n}$ (the latter set has cardinality n) and let $\gamma \in G$ be the product of the (disjoint) 2-cycles (a, b) such that $f(a) = b$. That is, γ is defined by

$$\gamma(a) = \begin{cases} f(a) & \text{if } a \in \bigcup_{j=1}^{j_0} A_{k',j} \\ b & \text{if } a = f(b) \\ a & \text{otherwise} \end{cases}$$

The permutation γ is in $G_{k'}$, and hence is in $G_{(E)}$ since $k' \notin I$. Therefore γ fixes Y and so $\gamma(y) \in Y$. By our choice of E_0 , E_0 is a support of $\gamma(y)$. To arrive at a contradiction and complete our proof of the sublemma we will show that the permutation $\eta = \gamma\phi\gamma^{-1}$ is in $G_{(E_0)}$ but that $\eta(\gamma(y)) \neq \gamma(y)$. The latter of these two assertions is the easiest: $\eta(\gamma(y)) = \gamma\phi\gamma^{-1}\gamma(y) = \gamma\phi(y)$. Since $\phi(y) \neq y$ and γ is a one-to-one function on \mathcal{N} , $\gamma(\phi(y)) \neq \gamma(y)$. For the proof that η is in $G_{(E_0)}$, let $Q \in E_0$. If $Q \notin \mathbf{P}_{k'}$, then η clearly fixes Q since $\eta \in G_{k'}$, so assume $Q \in E_0 \cap \mathbf{P}_{k'}$. Thus, as noted above, $A_{k',n} \in Q$. But ϕ only moves atoms in $\bigcup_{j=1}^{j_0} A_{k',j}$, and it follows that $\gamma\phi\gamma^{-1}$ only moves atoms in $\gamma\left(\bigcup_{j=1}^{j_0} A_{k',j}\right) \subseteq A_{k',n}$. Hence η fixes Q . This completes the proof of the sublemma. \square

Assume $\phi \in G$. Then there are elements ϕ_1 and ϕ_2 of G such that $\phi = \phi_1 \circ \phi_2$ and such that ϕ_1 fixes $\bigcup_{i \notin I} A_i$ pointwise and ϕ_2 fixes $\bigcup_{i \in I} A_i$ pointwise. By the sublemma $\phi(y) = \phi_1(\phi_2(y)) = \phi_1(y)$. We may therefore conclude that

$$\{\phi(y) : \phi \in G_{(E)}\} = \{\phi(y) : \phi \in G_{(E)} \text{ and } \phi \text{ fixes } \bigcup_{i \notin I} A_i \text{ pointwise}\}.$$

Let H be the group of permutations in $G_{(E)}$ that fix $\bigcup_{i \notin I} A_i$ pointwise, so that $\{\phi(y) : \phi \in G_{(E)}\} = \{\phi(y) : \phi \in H\}$.

For each $i \in I$, since $\mathbf{P}_i \cap E \neq \emptyset$ we choose $Q_i \in \mathbf{P}_i \cap E$. By (*) there is $j_{Q_i} \in \omega$ such that $Q_i = D \cup \{A_{i,j} : j > j_{Q_i}\}$ and therefore any $\phi \in G_{(E)}$ must fix $\{A_{i,j} : j > j_{Q_i}\}$ pointwise. Now we consider E_0 . Applying (*) again, there is a k_i such that $\{A_{i,j} : j > k_i\} \subset \bigcap (E_0 \cap \mathbf{P}_i)$. We may

assume that $k_i \geq j_{Q_i}$. Define

$$B_1 = \bigcup_{\substack{i \in I \\ j > k_i}} A_{i,j} \quad \text{and} \quad B_2 = \bigcup_{\substack{i \in I \\ j \leq k_i}} A_{i,j}.$$

Since any $\phi \in G_{(E)}$ must fix $\{A_{i,j} : j > k_i\}$ pointwise for each $i \in I$, it follows that any $\phi \in H$ can be written as a composition $\phi = \phi_2 \circ \phi_1$ where $\phi_1 \in H$ moves only atoms in B_1 and $\phi_2 \in H$ moves only atoms in B_2 (that is, $\phi_1 \in H \cap \text{Sym}(B_1)$ and $\phi_2 \in H \cap \text{Sym}(B_2)$). For such ϕ_1 , we have $\phi_1 \in G_{(E_0)}$, which implies that $\phi(y) = \phi_2(\phi_1(y)) = \phi_2(y)$. It follows that

$$\{\phi(y) : \phi \in G_{(E)}\} = \{\phi(y) : \phi \in H\} = \{\phi(y) : \phi \in H \cap \text{Sym}(B_2)\}.$$

Since B_2 is finite, the set above is finite, and this completes the proof of the lemma. \square

To complete the proof of the theorem assume $X = \{Y_i : i \in \omega\}$ is a countable set in \mathcal{N} and that each $Y_i, i \in \omega$, is countable in \mathcal{N} . Let E be a support of the enumeration $i \mapsto Y_i$, then for every $i \in \omega$, E is a support of Y_i . We will argue that $\bigcup X = \bigcup_{i \in \omega} Y_i$ is a countable union of finite sets in \mathcal{N} .

For each $y \in \bigcup X$, let $\text{Orb}_E(y) = \{\phi(y) : \phi \in G_{(E)}\}$. Every y in $\bigcup X$ is in the well-orderable set Y_i for some i , so by the lemma, each $\text{Orb}_E(y)$ is finite. Let $W = \{\text{Orb}_E(y) : y \in \bigcup X\}$. Clearly $\bigcup W = \bigcup X$. Since every element of W has support E , W is well orderable in \mathcal{N} . Therefore, since W is countable in the ground model, W is countable in \mathcal{N} . Hence $\bigcup X$ is a countable union of finite sets in \mathcal{N} . \square

4 Families of Finite Sets

In this section, we establish some results related to forms such as $\text{UT}(\aleph_0, < \aleph_0, \aleph_0)$ and $\text{UT}(\text{WO}, n, \text{WO})$ (forms 10 and 422(n)) and some of their consequences. We first observe some equivalences of form 10, $\text{C}(\aleph_0, < \aleph_0)$. Consider the statements:

- $\text{C}(\aleph_0, E)$: Every denumerable family of pairwise disjoint sets with an even number of elements has a choice function.
- $\text{PC}(\aleph_0, E)$: Every denumerable family of pairwise disjoint sets with an even number of elements has an infinite subset with a choice function.

Theorem 4.1.

- (i) $\text{C}(\aleph_0, E)$ iff $\text{PC}(\aleph_0, E)$.
- (ii) $\text{PC}(\aleph_0, E)$ iff 10.

Proof. (i) Clearly $\text{C}(\aleph_0, E)$ implies $\text{PC}(\aleph_0, E)$. For the converse fix a family $\mathcal{A} = \{A_i : i \in \omega\}$ of disjoint non-empty even numbered sets. Then the family $\mathcal{B} = \{B_i = \prod_{j \leq i} A_j : i \in \omega\}$ is a disjoint family of even numbered sets. It is easy to see that any partial choice function on \mathcal{B} yields a choice set for \mathcal{A} .

(ii) It suffices to show that $C(\aleph_0, E)$ implies form 10 because the converse is clear. Fix a family $\mathcal{A} = \{A_i : i \in \omega\}$ of disjoint non-empty finite sets. Then $\mathcal{B} = \{B_i = A_i \times 2 : i \in \omega\}$ is a pairwise disjoint family of even numbered sets. Thus $C(\aleph_0, E)$ implies that \mathcal{B} has a choice set C . Using C one can easily construct a choice set for the family \mathcal{A} . \square

Let $MC(\aleph_0, E, 2)$ and $PMC(\aleph_0, E, 2)$ stand for $C(\aleph_0, E)$ and $PC(\aleph_0, E)$ with the requirement of choosing a single element eased to that of choosing ≤ 2 elements.

Theorem 4.2.

- (i) $MC(\aleph_0, E, 2)$ iff $PMC(\aleph_0, E, 2)$.
- (ii) $MC(\aleph_0, E, 2)$ iff $MC(\aleph_0, < \aleph_0, 2)$.

Proof. (i) Clearly $MC(\aleph_0, E, 2)$ implies $PMC(\aleph_0, E, 2)$ For the converse, fix a family $\mathcal{A} = \{A_i : i \in \omega\}$ of disjoint non-empty even numbered sets. Then the family $\mathcal{B} = \{B_i = \prod_{j \leq i} A_j : i \in \omega\}$ is a pairwise disjoint family of even numbered sets. Let $\mathcal{C} = \{C_{n_i} : i \in \omega\}$ be a family of ≤ 2 -element sets such that $C_{n_i} \subset B_{n_i}$. Using \mathcal{C} , one can easily construct a family $\mathcal{D} = \{D_i : i \in \omega\}$ such that for all $i \in \omega$, $0 < |D_i| < 3$ and $D_i \subset A_i$.

The proof of (ii) is similar to the proof of (ii) in Theorem 4.1. \square

Corollary. $PMC(\aleph_0, E, 2) + 18$ iff $C(\aleph_0, E)$.

At this point we turn to finite choice axioms such as $374(n)$, $47(n)$, and $288(n)$ for various integers n (these are $UT(\aleph_0, n, \aleph_0)$, $C(WO, n)$, and $C(\aleph_0, n)$, respectively). Note that the figure in the introduction does not give any information on when, for example, $47(n)$ implies $47(m)$. However, much is known about the relationships between finite axioms of choice such as $47(n)$. See Note 15 in [6] for a long summary with references, and see also [1].

Theorem 4.3.

- (i) For every $n, m \in \mathbb{N}$, $288(nm)$ implies $288(n) + 288(m)$
- (ii) For every $n, m \in \mathbb{N}$, $288(n^m)$ implies $288(n) + 288(n^2) + \dots + 288(n^{m-1})$.
- (iii) For every $n, m \in \mathbb{N}$, $373(nm)$ implies $373(n) + 373(m)$

Proof. (i) Fix $\mathcal{A} = \{A_i : i \in \mathbb{N}\}$, a pairwise disjoint family of n -element sets. For every $j \in \mathbb{N}$ put $B_j = A_j \times m$. Clearly $\mathcal{B} = \{B_i : i \in \mathbb{N}\}$ is a pairwise disjoint family of mn -element sets. Thus, by $288(nm)$, \mathcal{B} has a choice set b . On the basis of b one can readily define a choice set for \mathcal{A} . Similarly, one can show that $288(nm)$ implies $288(m)$.

Now (ii) follows easily from (i), and (iii) is proved by an argument similar to that in (i). \square

Corollary. $288(12)$ implies $288(2) + 288(3) + 288(4) + 288(6)$.

Theorem 4.4.

- (i) There is a model of ZF^0 in which form $47(4)$ is true, and form $111=422(2)$ is true (and hence $374(2)$ is true), while $373(3)$ is false (and hence the forms $288(3n)$ are false). In particular, 111 does not imply $288(6)$.
- (ii) There is a model of ZF^0 in which the forms $47(2m+1)$ are true, while form $18=373(2)$ is false (and hence $288(2m)$ are false). In particular, $288(3)$ does not imply $288(2m)$ for any $m \in \omega$, $m \geq 1$.

Proof. (i) In the model $\mathcal{N}2^*(3)$ of [6], the set of atoms $A = \bigcup B$, where B is a countable set of pairwise disjoint 3 element sets, $T_i = \{a_i, b_i, c_i\}$. For each $i \in \omega$ define a function $\eta_i : T_i \rightarrow T_i$ such that $\eta_i : a_i \mapsto b_i \mapsto c_i \mapsto a_i$. \mathcal{G} is the group of all permutations ϕ of A such that for each $i \in \omega$, $\phi \upharpoonright T_i$ is either the identity, η_i , or η_i^2 . The set of supports is the set of all finite subsets of A . In this model there is a denumerable family of triples which has neither a choice function nor partial choice function. Thus 373(3) is false. It follows that 288(3) is false, and using Theorem 4.3 it follows that 288(3n) is false for every natural number n .

It is known that form 333 (MC(∞ , ∞ , odd): for any family of sets there is a function f which chooses an odd number of elements from each set in the family) is true in $\mathcal{N}2^*(3)$ (See [6]). It follows that 422(2) (equivalent to 111, C(WO,2)) holds in this model, and also that 47(4) (C(WO,4)), holds. This proves (i).

(ii) In the Second Fraenkel model, Model $\mathcal{N}2$ in [6], A is the set of atoms and B is a countable partition of A into pairs. \mathcal{G} is the group of permutations of A that leaves B pointwise fixed, and the set of supports is the set of all finite subsets of A . The set B has no partial choice function. Thus, form 374(2) fails in $\mathcal{N}2$. On the other hand, the even multiple choice axiom, MC(∞ , ∞ , even), form 334 in [6], holds in $\mathcal{N}2$. It is easy to see that for every $m \in \mathbb{N}$, 334 implies 47(2m + 1). Hence, 47(2m + 1) implies neither 373(2) nor 288(2m). \square

Lemma. *Assume that \mathcal{N} is the Fraenkel-Mostowski model determined by the set A of atoms, the group G of permutations of A and the filter \mathfrak{F} of subgroups of G . Assume $X \in \mathcal{N}$ has stabilizer subgroup $G_X \in \mathfrak{F}$. Then X has a choice function in \mathcal{N} if and only if there is a group $H \in \mathfrak{F}$ such that for every $y \in X$, there is a $t \in y$ such that for all $\phi \in H \cap G_X$, $\phi(y) = y$ implies $\phi(t) = t$.*

Proof. Assume the hypotheses. If X has a choice function $f \in \mathcal{N}$ with support group $H \in \mathfrak{F}$, then for every $y \in X$, $t = f(y)$ has the property that $\forall \phi \in H$, if $\phi(y) = y$, then $\phi(t) = t$.

For the converse, assume that $H \in \mathfrak{F}$ has the property that for every $y \in X$, there is an element $t \in y$ such that $\forall \phi \in H$, if $\phi(y) = y$ then $\phi(t) = t$. Let $K = H \cap G_X$; we'll write X as the disjoint union of its K -orbits. That is, $X = \bigcup \mathcal{O}$ where $\mathcal{O} = \{\text{Orb}_K(y) : y \in X\}$ and $\text{Orb}_K(y) = \{\phi(y) : \phi \in K\}$. For each $U \in \mathcal{O}$, we choose $y_U \in U$ and $t_U \in y_U$ such that $\forall \phi \in H$, $\phi(y_U) = y_U$ implies $\phi(t_U) = t_U$. (It may be the case that neither the function $U \mapsto y_U$ nor the function $U \mapsto t_U$ is in \mathcal{N} .) Let $C_U = \{\phi((y_U, t_U)) : \phi \in K\}$. It is fairly straightforward to show that $\bigcup_{U \in \mathcal{O}} C_U$ is a choice function for X with support group K . \square

Theorem 4.5. *If $n \in \omega$ and k has no divisors less than or equal to n , then $C(\infty, n)$ is true in $\mathcal{N}2^*(k)$. ($\mathcal{N}2^*(3)$ is defined in Theorem 4.4.)*

Proof. Assume the hypotheses and assume that X is a set of n element sets in $\mathcal{N}2^*(k)$. Note that $\mathcal{N}2^*(k)$ is constructed via a group G such that for all $\phi \in G$, $\phi^k = 1_A$ (the identity permutation on A). We will show that $(\forall y \in X)(\forall t \in y)(\forall \phi \in G)$ if $\phi(y) = y$ then $\phi(t) = t$. Then $C(\infty, n)$ will follow from the lemma.

Towards a contradiction, assume that $y \in X$, $t \in y$ and $\phi_0(y) = y$ but that $\phi_0(t) \neq t$. Then, since $|y| = n$, there is some $d \in \omega$, $1 < d \leq n$, such that $\phi_0^d(t) = t$. We assume that d is the least such natural number. Dividing k by d gives $k = qd + r$ where q and r are natural numbers and $0 < r < d$

since d is not a divisor of k . As noted above, $\phi_0^k(t) = t$. On the other hand $\phi_0^k(t) = \phi_0^{qd+r}(t) = \phi_0^r(t)$ since $\phi_0^d(t) = t$. But $\phi_0^r(t) \neq t$ by our choice of d . This is a contradiction. \square

Corollary. *Form 422(n), $(\forall i, 2 \leq i \leq n)C(WO, i)$, does not imply form 423, $(\forall n \geq 2)C(\aleph_0, n)$.*

Proof. Theorem 4.5 gives us that for any finite subset $W \subset \omega$, $(\forall n \in W)C(\infty, n)$ does not imply $C(\aleph_0, k)$ if k is chosen so that every divisor of $k > 1$ is greater than every element of W . \square

All the consistency results in this section are transferable to ZF using the results of [11]; this follows from the fact that form 422(n) and all the forms implied by it are what Pincus calls term choice sentences. (Also see notes 18 and 103 in [6].)

5 Reference tables

The matrix below summarizes what is known about relationships between forms discussed in Sections 2 and 3, using notation mostly consistent with [6]. In the matrix, if there is a “1” at row m , column n it means that statement m implies statement n (in ZF^0); a “3” means that there is a Cohen model in which statement m is true and statement n is false, and a “5” means that there is a model of ZF^0 in which statement m is true and statement n is false. An entry in boldface means that a proof of the result may be found in this paper; proofs and/or references for other entries can be found in [6] or are clear.

	8	31	32	122	151	165	231	338	373	419	420	421
8	1	1	1	3	3	3	3	1	1	1	1	1
31	3	1	1	3	3	3	3	1	1	1	1	3
32	3	3	1	3	3	3	3	3	1	1	1	3
122	3	3	3	1	3	3	3	3	1	3	3	3
151	3	3	1	1	1	5	5	1	1	3	3	5
165	3	3	1	1	5	1	5	5	1	3	3	5
231	3	3	1	1	1	1	1	1	1	3	3	1
338	3	3	1	3	3	3	3	1	1	3	3	5
373	3	3	3	3	3	3	3	3	1	3	3	3
419	3	3	3	3	3	3	3	3	3	1	1	5
420	3	3	3	3	3	3	3	3	3	3	1	5
421	3	3	1	3	3	3	3	3	3	3	3	1

Many of the forms discussed in Section 4 depend on a parameter, and this makes it awkward to summarize the relationships between the forms with a matrix. However, hopefully the following reference matrix will be of some use. For the forms with a parameter n , the case $n = 2$ is often special, and will be ignored for purposes of filling in the matrix. For example, form 111 is equivalent to form 47(2). But for all $n \geq 3$, there is a model of ZF in which Form 111 is true while form 47(n) is false, and so there is a “3” in the table entry. (This particular independence result follows from the second theorem, attributed to Gauntt, in Note 15 of [6].)

As with the previous table, a “1” indicates that the implication holds; a “3” indicates that the implication does not hold and there is a model of ZF which witnesses this; a “5” indicates that the implication does not hold and there is a model of ZFA which witnesses this. A mixed entry such as “1/3” indicates that the implication holds for some $n \geq 3$, but is known to be false for other $n \geq 3$. An entry of “0” indicates that it is unknown whether the implication holds.

The entries in boldface follow from results in this paper. Proofs and/or references for the proofs for most of the other entries in the table can be found in Part V or Note 15 of [6]. Entries in italics represent previously known results which are not listed in Part V or Note 15 of [6]; references for these may be found following the table.

		8	10	18	$47_{(n)}$	80	111	122	$288_{(n)}$	$373_{(n)}$	$374_{(n)}$	$422_{(n)}$	423
	8	1	1	1	3	1	<i>3</i>	3	1	1	1	3	1
	10	3	1	1	3	1	3	3	1	1	1	3	1
$373(2) =$	18	3	3	1	3	0	3	3	3	3	3	3	3
	$47_{(n)}$	3	3	1/3	1	1/3	1/3	3	1	1	3	3	3
$374(2) = 288(2) =$	80	3	3	1	3	1	3	3	3	3	3	3	3
$422(2) = 47(2) =$	111	3	3	1	3	1	1	3	3	3	3	3	3
	122	3	1	1	1	1	1	1	1	1	1	1	1
	$288_{(n)}$	3	3	1/3	3	1/3	3	3	1	1	3	3	3
	$373_{(n)}$	3	3	1/3	3	0/3	3	3	0	1	3	3	3
	$374_{(n)}$	3	3	1	3	1	3	3	1	1	1	3	3
	$422_{(n)}$	3	<i>3</i>	1	1	1	1	3	1	1	1	1	3
	423	3	<i>3</i>	1	3	1	3	3	1	1	1	3	1

References for italicized entries:

$8 \not\rightarrow 111$: See model $\mathcal{M}47(2, M)$ in [6], or equivalently, Theorem 2.1(E) of [12].

$422(n) \not\rightarrow 10$: See model $\mathcal{N}6$ in [6].

$423 \not\rightarrow 10$: A proof and further references can be found in [7] Chapter 7.

The open question responsible for the zeros in the table above is whether form $373(n)$, $\text{PUT}(\aleph_0, n, \aleph_0)$, is equivalent to form $288(n)$, $\text{UT}(\aleph_0, n, \aleph_0)$. In other words: Is it provable in ZF that if every countable union of n -element sets has a countably infinite subset, then every countable union of n -element sets is countable? We conjecture that the answer is no.

The permutation model in which form $373(2)$, $\text{PUT}(\aleph_0, 2, \aleph_0)$, fails most obviously is the second Fraenkel model $\mathcal{N}2$. It may be tempting to think that the independence of $288(2)$ from $373(2)$ may be easily established by adding some ideal of infinite supports to $\mathcal{N}2$, in hopes that $\text{PUT}(\aleph_0, 2, \aleph_0)$ will become true (while leaving $\text{UT}(\aleph_0, 2, \aleph_0)$ false). However, we have found that $\text{PUT}(\aleph_0, 2, \aleph_0)$ is false in such modifications of $\mathcal{N}2$.

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