A Characterization of Permutation Models in Terms of Forcing

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Abstract

We show that if N and M are transitive models of ZFA such that $N \subseteq M$, N and M have the same kernel and same set of atoms, and $M \models$AC, then N is a Frankel-Mostowski-Specker (FMS) submodel of M if and only if M is a generic extension of N by some almost homogeneous notion of forcing. We also develop a slightly modified notion of FMS submodels to characterize the case where M is a generic extension of N not necessarily by an almost homogeneous notion of forcing.

1 Introduction

Permutation models are used to produce independence results in ZFA (ZF set theory with extensionality modified to allow a set A of atoms— for purposes of this article, a proper class of atoms is not allowed). Many statements which are independent of ZFA are nonetheless known to hold in all permutation models: Some follow from the fact that AC holds in the kernel of every permutation model (e.g., "The power set of a well-orderable set is well-orderable"), and statements such as SVC (see Blass [1]) follow essentially from choice in the kernel together with the fact that there is only a *set* of atoms. In these cases, the results can be cast as theorems of ZFA; e.g. ZFA $\vdash (\mathrm{AC})^{\mathrm{kernel}} \to$ SVC. Other sentences that hold in every permutation model do not obviously follow from any simple principle expressible in the language of ZFA, and are proved "externally", making explicit use of the existence of a permutation group and filter, as in Howard [6]. This paper will produce an infimum for the set of all sentences that hold in every permutation model; that is, a single principle expressible

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as a sentence ψ in the language of set theory, itself true in every permutation model, such that any sentence ϕ that holds in all permutation models is provable from ZFA+ψ.

The existence of this ψ follows from the main theorem of the paper, which states, roughly, that a transitive model of ZFA is a permutation model if and only if some generic extension of it with the same kernel satisfies AC (but see Theorem 4.1 for the precise statement). This characterization gives the ψ mentioned above, since forcing is definable in the language of set theory.

The usual notion of a permutation model, as in Jech [7] or Brunner [3], will be called an *FMS model*, after Fraenkel, Mostowski, and Specker, who pioneered the techniques. It turns out that the characterization in terms of forcing is more natural for a slightly more general notion, that of an *almost-FMS model*, which will be defined in Section 3, facilitated by a slightly unorthodox definition of *permutation model* in Section 2.

Some of our results hold only for *transitive* models of set theory. Transitivity should be understood, even if not explicitly stated, whenever the term "model" is used.

2 Permutation Models

We will mainly follow the terminology in [7], but our definition of "permutation model" will be slightly different.

Definition 2.1. Given a model M of ZFAC (always with set A of atoms), let G be a group of permutations of A. A permutation $g \in G$ is recursively extended to a function of all of M by $gx = \{ gy \mid y \in x \}$. If this g acts on M, for example if g is a member of M, then g is an automorphism of M. In general it will be an isomorphism from M to the image of M. For any $x \in M$, the stabilizer subgroup $\{ g \in G \mid gx = x \}$ of x will be denoted by G_x. A *normal filter on* G is a non-empty set of subgroups of G which is closed under supergroup, finite intersection, and conjugation, and which contains G_a for each atom $a \in A$. Given a normal filter $\mathcal{F}$, an element of $x \in M$ is called *symmetric (or* $(G, \mathcal{F})$*-symmetric)* whenever $G_x \in \mathcal{F}$; if x and all elements of x's transitive closure are symmetric, then x is *hereditarily symmetric*. If the class of hereditarily symmetric elements of M is a model of ZFA, it is called a *permutation model*; it is the *permutation submodel of* M determined by G and $\mathcal{F}$.

Given x and y in M, we say that x *supports* y when the pointwise stabilizer of x is a subset of G_y. Thus if x is symmetric and $\{x\}$ supports y, then y is symmetric.

A *pure set* in a model M of ZFA is a set with no atoms in its transitive closure. The *kernel* M is the class of M's pure sets and is denoted M_{ker}. Clearly M has the same kernel as any permutation submodel of M.

The definition of permutation model above departs from standard practice in one important respect: In a standard presentation, it is usually understood that G and $\mathcal{F}$ are members of M. Here we have deliberately left out that restriction. The normal practice of assuming G and $\mathcal{F}$ are members of M is useful in that the resulting class of hereditarily symmetric elements is guaranteed to be a model of ZFA. There is no such guarantee if G and $\mathcal{F}$ are recklessly chosen outside of M. In practice we will want the hypotheses of the following theorem to hold.

Theorem 2.2. *Given $M \models \text{ZFAC}$ and G as above, and a normal filter $\mathcal{F}$ on G, let $N \subseteq M$ be the class of hereditarily symmetric members of M. If (i) $gn \in M$ for all $g \in G$ and all $n \in N$, and (ii) N is a class for M, then G acts on N by automorphisms and $N \models \text{ZFA}$.*

Proof (sketch). Hypotheses (i) and (ii) are what are needed to carry out a proof like that in [7, Thm 4.1], which was written for the case where G and $\mathcal{F}$ are in M. Hypothesis (i) is used to show almost universality of N in M as follows:

It is necessary to show that each initial segment $M_\alpha \cap N$ is symmetric. We prove by induction on α that $M_\alpha \cap N$ is stabilized by all of G. For successor stages, assume $M_\alpha \cap N$ is stabilized by all of G and let $x \in M_{\alpha+1} \cap N$. Given any $g \in G$, we have $gx \in M_{\alpha+1}$, by hypothesis (i) and the fact that g preserves rank. To see that $gx \in N$, $x \subset M_\alpha \cap N$, so by induction $gx \subset M_\alpha \cap N \subset N$. Also, gx is symmetric, since x is, so $gx \in N$. The induction at limit stages is clear. □

Example 2.3. Let $M \models \text{ZFAC} + $ "A is uncountable". We will give an example of a permutation submodel of M which is not an FMS submodel of M. In M, fix a partition P of A into infinite sets such that infinitely many of the parts in P are countable, and infinitely many are uncountable. Let C be the set of countable members of P. Let M^+ be a generic extension of M in which the he cardinality of A is $\aleph_0$. In M^+, let G be the set of permutations of A that stabilize P, and let $\mathcal{F}$ be the *finite support* filter on G, that is, the filter on G generated by $\{ G_{\langle a_1,\dots a_n\rangle} \mid a_1, \dots a_n \in A \}$. Let N be the set of hereditarily symmetric elements of M. It will follow from Theorem 3.5 below that the hypotheses of Theorem 2.2 are satisfied, so that $N \models \text{ZFA}$.

Each $p \in P$ is symmetric; in fact, if $a \in p$ then $\{a\}$ supports p. To prove this, suppose $g \in G_a$. Assuming $a \in p \in P$ yields $a = ga \in gp \in gP = P$ (every $g \in G$ stabilizes P). But p is the only member of P that has a as a member, since P is a partition, so $gp = p$.

Each $p \in P$ is thus hereditarily symmetric (since each is a subset of A), so is in N. Obviously P is symmetric, since every $g \in G$ fixes P, and we just showed that all the members of P are in N, so P is also in N. On the other hand, C is not in N.

To see that C does not have a finite support, let S be a finite set of atoms, and choose $p_0 \in C$ and $p_1 \in P \setminus C$ such that p_0 and p_1 are disjoint from S. Since p_0 and p_1 are both countable in M^+, there is a permutation $g \in G$ which switches p_0 and p_1 while fixing every atom not in p_0 or p_1. Such a g fixes S pointwise, but does not fix C. Thus S does not support C.

The reason N is not an FMS submodel of M is that any FMS submodel of M which has P also has C. To see this, let G' and $\mathcal{F}'$ define an FMS submodel of M

such that P is hereditarily $(G', \mathcal{F}')$-symmetric. Then $G'_P \in \mathcal{F}'$. If $g \in G'_P$, then g is an automorphism of M fixing P, so it sends countable members of P to countable members of P (here we mean countable in M, of course); in other words, $gC = C$. Thus $G'_C \supseteq G'_P$, so C is symmetric. Then since $C \subset P$ and P is assumed to be hereditarily symmetric, C is hereditarily symmetric.

Here are two well-known facts regarding group/filter pairs that are equivalent in the sense of determining the same permutation submodels of a given model of ZFAC.

Lemma 2.4. *Let N be the permutation submodel of M determined by G and $\mathcal{F}$.*

(a) *If $\mathcal{F}_1 = \{ F \in \mathcal{F} \mid (\exists x \in N)\, F \supset G_x \}$, then $(G, \mathcal{F})$-symmetry is the same as $(G, \mathcal{F}_1)$-symmetry.*

(b) *Let $G_1 < G$; let $\mathcal{F}_1 = \{ F \cap G_1 \mid F \in \mathcal{F} \}$. If $(\forall n \in N)(\forall g \in G)(\exists g_1 \in G_1)\, g_1 n = gn$, then $(G, \mathcal{F})$-symmetry is the same as $(G_1, \mathcal{F}_1)$-symmetry.*

Lemma 2.4(b) is usually stated in terms of topological groups ("A dense subgroup induces the same symmetry structure"), see Brunner and Rubin [4, 2.7].

Lemma 2.4(a) allows us to assume without loss of generality that every group in $\mathcal{F}$ contains a stabilizer subgroup of some member of N. Notice furthermore that if $\mathcal{F}$ has this property, then there is some set $B \in N$ such that $\mathcal{F} = \{ F < G \mid (\exists x \in B)\, F \supset G_x \}$. Thus every permutation model is determined by a group G and what we will call a *normal base* (a slight modification of the concept of *normal ideal*, see [7]):

Definition 2.5. A *normal base* for N as the permutation submodel of M determined by G and $\mathcal{F}$ is a set $B \in N$ such that: (1) $G_B = G$, (2) every element of N is supported by some $\{b\}$ with $b \in B$, and (3) if b_1 and b_2 are in B, then there is a $c \in B$ such that $G_c = G_{b_1} \cap G_{b_2}$.

For example, there will always be some sufficiently large initial segment N_α of N that is a normal base. If $\mathcal{F}$ is the finite support filter on G, then the set of finite sequences of atoms is a normal base.

3 FMS Submodels and Almost-FMS Submodels

The idea of forcing over a model of ZF may be easily generalized to work in a model of ZFA. The main question that needs to be answered is: What serves as names for atoms? Following Blass and Scedrov (see [2] for more details), we will take the atoms to be names for themselves. The rest of the names are defined recursively as usual in ZF: The class $V^{\mathbf{P}}$ of $\mathbf{P}$-names for a notion of forcing $\mathbf{P}$ are the subsets of $V^{\mathbf{P}} \times \mathbf{P}$. For any generic filter Γ, define for each atom a $\mathsf{Val}_\Gamma(a) := a$, and for each set x $\mathsf{Val}_\Gamma(x) := \{ \mathsf{Val}_\Gamma(y) \mid (\exists p \in \Gamma) \langle y, p \rangle \in x \}$, as usual in ZF.

Recall that a notion of forcing $\mathbf{P}$ is called *almost homogeneous* whenever for any two conditions p and q in $\mathbf{P}$, there is an automorphism σ of $\mathbf{P}$ such that $\sigma p \parallel q$. The following lemma is well-known.

Lemma 3.1. *If $\mathbf{P}$ is almost homogeneous, and $p \Vdash \phi(\check{x})$ for some $p \in \mathbf{P}$ and some x in the ground model, then $\Vdash_{\mathbf{P}} \phi(\check{x})$.*

Definition 3.2. A set x is *almost in M* whenever (i) M^+ is a generic extension of M by an almost homogeneous notion of forcing in M and (ii) x is definable in M^+ using only parameters in M. Similarly, a class X which is a definable class in M^+ is *almost in M* if it is definable using only parameters in M. Observe that if y is definable in M^+ using only a parameter that is almost in M, then y is also almost in M.

The "almost in" concept will be useful through the following lemma, whose proof is straightforward from Lemma 3.1:

Lemma 3.3. *If $x \subseteq M$ is almost in M, then x is a definable subclass of M. Hence, if x is almost in M and x is a subset of some member of M, then $x \in M$.*

Definition 3.4. An *FMS submodel* of M is a permutation submodel of M determined by some group G and normal filter $\mathcal{F}$ both in M. An *almost-FMS submodel* of M is a permutation submodel of M obtained by some G almost in M and a normal filter $\mathcal{F}$ on G.

The definition of almost-FMS submodel does not require $\mathcal{F}$ to be almost in M, but doing so would entail no loss of generality: If G is almost in M and $\mathcal{F}$ is not, then $\mathcal{F}$ may be replaced with F_1 (as defined in Lemma 2.4(a)) without changing the permutation submodel determined, and this $\mathcal{F}_1$ is almost in M.

In Example 2.3 above, N is an almost-FMS submodel of M if the forcing notion making A countable in the extension is taken to be almost homogeneous. The group G is definable in the generic extension M^+ using only the parameter $P \in M$, so that G is almost in M. To see that this N is a model of ZFA, apply the following lemma, taking S to be the set of finite subsets of A.

Theorem 3.5. *Let G be almost in $M \models$ ZFAC, and let $S \in M$. If G acts on S and $\mathcal{F} := \{\, F < G \mid (\exists s \in S)\, G_s \subseteq F \,\}$ is a normal filter on G, then the class N of hereditarily symmetric elements of M is a model of* ZFA*, and hence is an almost-FMS submodel of M.*

Proof. Since the hypotheses imply that $\mathcal{F}$ is almost in M, it is easy to see using Lemma 3.3 that the hereditarily symmetric elements form a class for M Thus hypothesis (ii) of Theorem 2.2 is satisfied, and it just remains to check that hypothesis (i) is satisfied: Let $x \in N$; we want to show that $gx \in M$ for all $g \in G$.

The proof is by $\in$-induction, so assume, by the inductive hypothesis, that $gy \in M$ for all $y \in x$; i.e., $gx \subseteq M$. From the definition of $\mathcal{F}$, there is an $s \in S$ such that $\{s\}$

supports x. Then $\{gs\}$ supports gx, and it follows that for any $h \in G$, if $hs = gs$, then $hx = gx$. Thus,

$$\forall z\, [z = gx \;\leftrightarrow\; \exists h \in G\, (hs = gs \,\wedge\, z = hx)\,].$$

This is a definition of gx using G, s, gs, and x as parameters, and these are all in M or almost in M ($gs \in M$ since G acts on S and hence $gs \in S \in M$). Therefore gx is almost in M, and since $gx \subseteq M$, gx is really in M.

Theorem 2.2 may now be applied to show that $N \models$ ZFA. □

4 The Generator Poset

The main theorem may now be stated in terms of FMS and almost-FMS models.

Main Theorem 4.1. *Let $N \subseteq M$ be transitive models of* ZFA *with the same kernel and same set of atoms, and with $M \models$* AC*.*

(a) *N is an almost-FMS submodel of $M \iff M$ is a generic extension of N.*

(b) *N is an FMS submodel of $M \iff M$ is a generic extension of N by some almost homogeneous notion of forcing.*

Remark. In fact, if N is the almost-FMS model determined by G and $\mathcal{F}$, and N^+ is the extension determined by G and the filter generated by $\mathcal{F} \cup \{H\}$ for some subgroup H of G, then N^+ is a generic extension of N. This generalization of the $\Rightarrow$ direction of Theorem 4.1 (a) has been previously proved (for FMS models) by Andreas Blass (personal communication). The proof we give of that direction of part (a) is essentially the same as Blass', but with the point of view changed somewhat to facilitate the remaining proofs.

We will establish some notation and definitions to be used throughout this section, as we prove Theorem 4.1 in several steps. We first work towards a proof of the $\Rightarrow$ direction (of both parts (a) and (b) simultaneously), so let N be the almost-FMS submodel of M determined by G and $\mathcal{F}$. Let G and $\mathcal{F}$ be members of M^+, a generic extension of M by some almost homogeneous notion of forcing (where possibly $M^+ = M$, in which case we are in the situation of part (b)). Using Lemma 2.4(a), assume without loss of generality that every element of $\mathcal{F}$ contains the stabilizer of some element of N.

A key observation is that because $M \models$ AC and N has the same kernel as M, N has as much information as M in some sense. To make this precise, we shall construct an isomorphic copy M^* of M inside it's own kernel M_{ker}; this copy will also be in N_{ker}. Let κ be the cardinality of A in M, and let $A^* = \kappa \times \{0\}$. Define recursively

$$M^* = A^* \cup \{\, \langle s, 1\rangle \;\mid\; (\forall t \in s)\, t \in M^* \,\},$$

and define a membership relation E on M^* by

$$\langle x, i\rangle \mathrel{\mathsf{E}} \langle s, 1\rangle \quad :\Leftrightarrow \quad \langle x, i\rangle \in s.$$

It is not difficult to see that $\langle M, \in\rangle \cong \langle M^*, \mathsf{E}\rangle$ (hereafter, we will usually simply write $M \cong M^*$). In particular, each bijection $b : A^* \to A$ in M extends uniquely to an isomorphism $b : M^* \to M$ by

$$b(\langle s, 1\rangle) = \{\, b(t) \mid t \in s \,\}.$$

Hereafter, bijections $A^* \to A$ will automatically be understood to also be functions on M^* in this way.

This trick of building a copy of M inside its kernel is borrowed from [2], where it was used to prove

Lemma 4.2. *If $M \models ZFAC$ and $b : \kappa \to A$ is a bijection in M for some cardinal κ, then M is generated by its kernel and b.*

Notation. Fix once and for all an isomorphism $j : M^* \to M$ in M. Let J be the orbit $\{\, gj \mid g \in G \,\}$ of j. Let B be a normal base (Def. 2.5) for N, so that $\mathcal{F} = \{\, F < G \mid (\exists b \in B)\, F \supset G_b \,\}$. For notational convenience later, choose B such that $A \subset B$.

We will use priming to denote j^{-1}; that is, for each $x \in M$, $x' := j^{-1}(x)$. In keeping with the notations M^* and A^*, let N^* and B^* stand for $j^{-1}``N$ and $j^{-1}``B$, respectively.

Definition 4.3. The *generator poset for N and M (based on B and J)* is a notion of forcing $\mathbf{P} = \mathbf{P}(B, J)$ defined as follows. The conditions of $\mathbf{P}$ are of the form $\langle b', gb\rangle$ for each $b \in B$ and $g \in G$ (equivalently, $\langle z, i(z)\rangle$ for each $z \in B^*$ and $i \in J$), and the order relation is defined by letting $\langle z_0, x_0\rangle \le \langle z_1, x_1\rangle$ if and only if every isomorphism $N^* \to N$ in J sending z_0 to x_0 also sends z_1 to x_1 (equivalently, $\langle b', gb\rangle \le \langle c', hc\rangle$ iff $g^{-1}h \in G_c$ and $G_b \subseteq G_c$). Note that $\mathbf{P}$ may not be a partial order in the strict sense; i.e. $p \le q \le p$ does not necessarily imply $p = q$.

It will be technically convenient to sometimes use the abbreviation P^g_F for $\langle b', gb\rangle$ when $F = G_b$. This notation is not strictly well-defined, since there may be more than one element of $\mathbf{P}$ that P^g_F refers to, e.g. when $F = G_b = G_c$ and $\langle b', gb\rangle$ and $\langle c', gc\rangle$ are both in $\mathbf{P}$. But in that case, we have $\langle b', gb\rangle \le \langle c', gc\rangle \le \langle b', gb\rangle$, so the ambiguity is irrelevant as far as forcing is concerned. Observe that

$$P^g_F \le P^h_E \quad \Leftrightarrow \quad g^{-1}h \in E \text{ and } F \subseteq E \tag{$*$}$$

and that the maximal element of $\mathbf{P}$ is $\mathbf{1_P} = P^g_G$ (for any $g \in G$). It is clear from the definition of $\mathbf{P}$ that if P^g_F is in $\mathbf{P}$, then P^h_F is also in $\mathbf{P}$ for any $h \in G$. It follows from the fact that B is a normal base that whenever P^g_F and P^g_E are in $\mathbf{P}$, so is $P^g_{F\cap E}$.

Lemma 4.4. *The generator poset $\mathbf{P}(B, J)$ for N and M is in N, and is stabilized by all of G.*

Proof. The definition of $\mathbf{P}$ may be given using as parameters only elements of M and the set J, which is almost in M. So $\mathbf{P}$ is almost in M. And clearly $\mathbf{P} \subseteq M$, so by Lemma 3.3, it is in M. The set of all conditions is clearly stabilized by all of G, and each condition $\langle b', gb \rangle$ is in N (since $B \subset N$ and G acts on N), so the set of all conditions is in N.

It just remains to check that $G_{\leq} = G$, where $\leq$ is the order relation on $\mathbf{P}$. Let $\gamma \in G$. Note that $\gamma P^g_F = P^{\gamma g}_F$. For any E and F in $\mathcal{F}$ with $E \subseteq F$, using $(*)$ we have

$$P^g_F \leq P^h_E \;\Leftrightarrow\; g^{-1}h \in E \;\Leftrightarrow\; (\gamma g)^{-1}\gamma h \in E \;\Leftrightarrow\; \gamma P^g_F \leq \gamma P^h_E,$$

which finishes the proof. □

The idea now is to show that M is a $\mathbf{P}$-generic extension of N, where $\mathbf{P}$ is the generator poset. The set of conditions in $\mathbf{P}$ is a subset of $B^* \times B$. It will turn out that a generic filter $\Gamma \subset \mathbf{P}$ is actually a one-to-one function $B^* \to B$ which extends in a definite way to an isomorphism $f : N^* \to N$ (if B is transitive, then Γ is an isomorphism $\langle B^*, \mathsf{E} \rangle \to \langle B, \in \rangle$). Thus the conditions of $\mathbf{P}$ can be interpreted as specifying partial isomorphisms.

We need a $\mathbf{P}$-name $\dot{f}$ for this isomorphism $f : N^* \to N$ which appears in the extension. Technically, we'll write $\dot{f}$ as a set name for the bijection $A^* \to A$ that determines the isomorphism:

Notation. Given a generator poset $\mathbf{P}$, define $\dot{f} \;:=\; \{\, \langle \check{p}, p \rangle \mid p \in \mathbf{P} \cap (A^* \times A) \,\}$.

Lemma 4.5. *Using the notation and definitions in this section,*

(a) $\Vdash_{\mathbf{P}} \dot{f}$ *is a bijection* $(A^*)\check{} \to \check{A}$.

(b) M *is a* $\mathbf{P}$*-generic extension of* N.

Proof. (a) Fix a generic $\Gamma \subset \mathbf{P}$, and let $f := \mathsf{Val}_\Gamma(\dot{f})$. It is immediately clear that f is a subset of $A^* \times A$. Now given any $P^g_F \in \mathbf{P}$, we have $P^g_{F \cap E} \leq P^g_F$ (for any $E \in \mathcal{F}$), and $P^g_{F \cap E} \leq \langle z, a \rangle$ when $E = G_{j(z)}$ and $a = g(j(z))$. Thus for any $z \in A^*$ there is $a \in A$ such that $\langle z, a \rangle \in \Gamma$, and vice versa; hence, $\mathsf{Dom}(f) = A^*$ and $\mathsf{Ran}(f) = A$. To see that f is a function and that it is one-to-one, observe that P^g_F extends both $\langle z, a \rangle$ and $\langle y, b \rangle$ if and only if $F \subseteq G_{j(z)} \cap G_{j(y)}$ and $g(j(z)) = a$ and $g(j(y)) = b$. Such an F always exists, but such a g exists only if $(z = y \wedge a = b) \vee (z \neq y \wedge a \neq b)$.

(b) The generic set $\Gamma_0 \subset \mathbf{P}$ which yields M as a $\mathbf{P}$ generic extension is

$$\Gamma_0 := \{\, P^1_F \in \mathbf{P} \mid F \in \mathcal{F} \,\} = \{\, \langle b', b \rangle \mid b \in B \,\}.$$

Γ_0 is a filter because $P^1_{E \cap F} \in \Gamma_0$ extends both P^1_E and P^1_F, and because if $P^1_F \leq P^g_E$ then $g \in E$ (by $(*)$ in definition 4.3), and thus $P^g_E = P^1_E \in \Gamma_0$.

To prove that Γ_0 is generic over N, let $D \in N$ be dense in $\mathbf{P}$, and let $F \subseteq G_D$ with $P^1_F \in \mathbf{P}$. Since D is dense, there is $P^g_E \in D$ with $P^g_E \leq P^1_F$. By $(*)$ we have $g^{-1} \in F$. Since F stabilizes D, we have $P^1_E = g^{-1} P^g_E \in D$, witnessing that $D \cap \Gamma_0$ is non-empty.

Observe that if Γ_0 is defined as above, then $\mathsf{Val}_{\Gamma_0}(\dot{f}) = j$. This is because for each $a \in A$, $P^1_{G(a)} = \langle a', a \rangle \Vdash \dot{f}(a') = \check{a}$ (and recall that $a' := j^{-1}(a)$). Since M is generated by its kernel and j, we know that $M \subseteq N[\Gamma_0]$. We also have $N \subseteq M$ and $\Gamma_0 \in M$, so $N[\Gamma_0] = M$. □

So far, the ⇒ direction of Theorem 4.1, part (a) has been proved. The ⇒ direction for part (b) follows from the next lemma.

Lemma 4.6. *If N is an FMS submodel of M, then the generator poset $\mathbf{P}$ is almost homogeneous.*

Proof. Since N is an FMS submodel of M, we may assume that G and $\mathcal{F}$ are in M. Using the fixed isomorphism $j : M^* \to M$, each automorphism g of M has an analogue $j^{-1} \circ g \circ j$ which is an automorphism of M^*. Let $G^* = \{ j^{-1}gj \mid g \in G \}$. G^* is in M, and since it is in the kernel, it is also in N. Since G acts by automorphisms on N, G^* acts by automorphisms on N^*.

The idea is that each automorphism $h^* \in G^*$ of N^* induces an automorphism $\tilde{h}$ of $\mathbf{P}$ by $\tilde{h} : \langle s', gs \rangle \mapsto \langle h^*s', gs \rangle$. First, we'll check that $\tilde{h}\langle s', gs \rangle$ is really in $\mathbf{P}$. Since $h^* \in G^*$, we have $h = j \circ h^* \circ j^{-1} \in G$. Recall that G acts on the normal base B, so $hs \in B$ when $s \in B$ and $h \in G$. Thus, by definition of $\mathbf{P}$, $\langle (hs)', k(hs) \rangle$ is an element of $\mathbf{P}$ for any $k \in G$. Take $k = gh^{-1}$, and observe that $(hs)' = h^*s'$. Thus $\langle h^*s', gh^{-1}(hs) \rangle = \langle h^*s', gs \rangle$ is in $\mathbf{P}$, and this is what we where checking. Essentially, $\tilde{h}$ is a permutation of $\mathbf{P}$ because h^* is a permutation of B^* (because h is a permutation of B).

To see that each $\tilde{h}$ respects the order relation, suppose $\langle s', x \rangle \leq \langle t', y \rangle$ in $\mathbf{P}$. We want to show that $\langle h^*s', x \rangle \leq \langle h^*t', y \rangle$; that is, to show that every $f \in J$ sending h^*s' to x also sends h^*t' to y. This is verified by the following computation:

$$f(h^*s') = x \;\to\; (f \circ h^*)s' = x \;\to\; (f \circ h^*)t' = y \;\to\; f(h^*t') = y.$$

This uses the fact that if $f \in J$, then $f \circ h^* \in J$. To see that this is true, let $f = gj$ with $g \in G$. Then $f \circ h^* = (gj) \circ (j^{-1}hj) = ghj$, which is in J since $gh \in G$.

So we know that N has a healthy supply of automorphisms of $\mathbf{P}$. Now given $\langle s', gs \rangle$ and $\langle t', ht \rangle$ in $\mathbf{P}$, we want a $\sigma \in \mathsf{Aut}(\mathbf{P})$ such that $\sigma\langle s', gs \rangle \parallel \langle t', ht \rangle$. Let $\sigma = (j^{-1} \circ (h^{-1}g) \circ j)\tilde{}$. Compute:

$$\sigma\langle s', gs \rangle = \langle j^{-1}h^{-1}gjs', gs \rangle = \langle j^{-1}(h^{-1}gs), gs \rangle = \langle (h^{-1}gs)', h(h^{-1}gs) \rangle = P^h_F,$$

where $F = G_{h^{-1}gs}$. Let $E = G_t$, so $\langle t', ht \rangle = P^h_E$. Now it is clear that P^h_F and P^h_E are compatible, since $P^h_{F \cap E}$ lies below both of them. □

The ⇒ directions of the Main Theorem 4.1 have now been proved. For the converses, it will help to assume that G has a sort of canonical form, which the next lemma will allow.

Lemma 4.7. *Let $H = (\mathsf{Aut}(N)_{\mathbf{P}})^{M^+}$ (that is, H is the group in M^+ of all automorphisms of N stabilizing $\mathbf{P}$). Let $\mathcal{E} = \{\, F < H \mid (\exists b \in B) F \supseteq H_b \,\}$.*

(a) *$(G, \mathcal{F})$-symmetry is the same as $(H, \mathcal{E})$-symmetry, so replacing $(G, \mathcal{F})$ by $(H, \mathcal{E})$ gives the same permutation submodel.*

(b) *The generator poset $\mathbf{P}$ (whose definition makes use of G and $\mathcal{F}$ implicitly) does not change when $(G, \mathcal{F})$ is replaced by $(H, \mathcal{E})$.*

(c) *For each $p \in \mathbf{P}$, there is a filter $\Gamma \subset \mathbf{P}$ generic over N such that $p \in \Gamma \in M^+$.*

(d) *If $G = H$, then $\{\, \mathsf{Val}_\Gamma(\dot{f}) \mid \Gamma \in M^+$ and Γ is $\mathbf{P}$-generic over $N \,\} = \{\, gj \mid g \in G \,\}$.*

Proof. (a) Observe that $G < H$ and $\mathcal{F} = \{\, E \cap G \mid E \in \mathcal{E} \,\}$. By Lemma 2.4(b), it suffices to show that given $n \in N$ and $h \in H$, there is a $g \in G$ such that $gn = hn$. If $h \in H$, then $h\mathbf{P} = \mathbf{P}$, so $h\langle n', n\rangle = \langle n', hn\rangle \in \mathbf{P}$. Thus $\langle n', hn\rangle = \langle n', gn\rangle$ for some $g \in G$, whence $gn = hn$.

(b) Replacing $(G, \mathcal{F})$ with $(H, \mathcal{E})$ does not entail any change in the normal base B. By the proof of part (a) $\{\, \langle b', gb\rangle \mid b{\in}B \wedge g{\in}G \,\} = \{\, \langle b', hb\rangle \mid b{\in}B \wedge h{\in}H \,\}$, so the set of conditions in the generator poset is unchanged.

Taking two arbitrary conditions $\langle b', gb\rangle$ and $\langle c', hc\rangle$ (where g and h are in G), we have $\langle b', gb\rangle \le \langle c', hc\rangle$ if and only if $g^{-1}h \in G_c$ and $G_b \subseteq G_c$. Replacing $(G, \mathcal{F})$ with $(H, \mathcal{E})$ does not change the order relation, since $g^{-1}h \in G_c$ iff $g^{-1}h \in H_c$, and $G_b \subseteq G_c$ iff $H_b \subseteq H_c$.

(c) Let $\langle b', gb\rangle$ be an arbitrary condition in $\mathbf{P}$. Let $\Gamma_0 = \{\, \langle b', b\rangle \mid b \in B \,\}$ and observe that $\langle b', gb\rangle = g\langle b', b\rangle \in g\Gamma_0$. By the proof of Lemma 4.5(b), $M = N[\Gamma_0]$. Since g is an automorphism of N and $g\mathbf{P} = \mathbf{P}$, $g\Gamma_0$ is also a $\mathbf{P}$-generic filter over N. And $g\Gamma_0$ is in M^+ since g and Γ_0 are in M^+.

(d) Let $g \in H$ and recall from the proof of Lemma 4.5(b) that $j = \mathsf{Val}_{\Gamma_0} \dot{f}$. Reasoning as in the proof of part (c), we have

$$gj = g\,\mathsf{Val}_{\Gamma_0} \dot{f} = \mathsf{Val}_{g\Gamma_0}(g\dot{f}) = \mathsf{Val}_{g\Gamma_0} \dot{f}.$$

Conversely, let $\Gamma \in M^+$ be $\mathbf{P}$-generic over N, and let $f = \mathsf{Val}_\Gamma \dot{f}$. Since f and j are both isomorphisms $N^* \to N$ in M^+, $f = gj$ where $g = f \circ j^{-1}$. It just remains to show that $g \in H$.

Claim: $\Vdash_{\mathbf{P}} \dot{f}(\check{\mathbf{P}}') = \check{\mathbf{P}}$.

Proof of Claim: We show that no condition in $\mathbf{P}$ forces $\dot{f}(\check{\mathbf{P}}') \neq \check{\mathbf{P}}$. Let $p = \langle b', hb\rangle$ be an arbitrary condition in $\mathbf{P}$. Then $p \in h\Gamma_0$, where $M = N[\Gamma_0]$. $\mathsf{Val}_{h\Gamma_0} \dot{f} = hj$, and $hj(\mathbf{P}') = h\mathbf{P} = \mathbf{P}$. Thus $p \nVdash \dot{f}(\check{\mathbf{P}}') \neq \check{\mathbf{P}}$.

By the claim, $f(\mathbf{P}') = \mathbf{P}$, so we have $g\mathbf{P} = fj^{-1}(\mathbf{P}) = f(\mathbf{P}') = \mathbf{P}$. And clearly $g = f \circ j^{-1}$ is in $\mathsf{Aut}(N)^{M^+}$, so $g \in H$. $\square$

Since M and N have the same kernel, forcing with $\mathbf{P}$ from N doesn't add pure sets, at least when the generic set is Γ_0. We can now see that forcing with $\mathbf{P}$ from N never adds pure sets, regardless of the generic set.

Corollary 4.8. *Any* $\mathbf{P}$*-generic extension* $N[\Gamma]$ *of* N *is isomorphic to* M *(in any model containing both* $N[\Gamma]$ *and* M*).*

Proof. By Lemma 4.7(a & b), we may assume without loss of generality that $G = (\mathsf{Aut}(N)_{\mathbf{P}})^{M^+}$.

First note that all $\mathbf{P}$-generic extensions of N that happen to be contained in M^+ are isomorphic: Let Γ_0 be a $\mathbf{P}$-generic filter such that $M = N[\Gamma_0]$. If $\Gamma \subset \mathbf{P}$ happens to be a member of M^+, then by Lemma 4.7(d) we have $\Gamma = g\Gamma_0$ for some $g \in G$. It follows that $\mathsf{Val}_{\Gamma_0}(x) \mapsto \mathsf{Val}_{\Gamma}(x)$ is an isomorphism $N[\Gamma_0] \to N[\Gamma]$. In particular, no $\mathbf{P}$-generic extension of N in M^+ adds new pure sets.

It follows from the above and Lemma 4.7(c) that no condition in $\mathbf{P}$ forces the existence of new pure sets, so in general $N[\Gamma]$ has the same kernel as N. Now using the two facts (1) $N[\Gamma]$ and M have the same kernel, and (2) by Lemma 4.5, $N[\Gamma]$ and M both have bijections from A^* (which is well-ordered) to A, we can produce an isomorphism $N[\Gamma] \to M$ by Lemma 4.2. □

The next theorem, besides being useful in finishing the proof of Theorem 4.1, shows why we call $\mathbf{P}$ the "generator" poset. Together with the kernel, $\mathbf{P}$ generates the permutation model.

Theorem 4.9. *If* N *is an almost-FMS submodel of* M *and* $\mathbf{P}$ *is the generator poset, then* N *is generated by its kernel* N_{ker} *and* $\langle \mathbf{P}, \leq \rangle$ *in the sense that there is a surjective map* $\Phi : N_{\mathrm{ker}} \times \mathbf{P} \twoheadrightarrow N$ *which is* Δ_1^{ZFA} *definable using only* $\langle \mathbf{P}, \leq \rangle$ *and pure sets as parameters.*

Remark. Theorem 4.9 may be thought of as a more precise version of a result in [3, Theorem 3.1]: If $S \in N$ is such that for every $x \in N$ there is $\alpha \in \mathrm{On}$ and a surjection $f : S \times \alpha \twoheadrightarrow x$, then there is a surjection $F : S \times \mathrm{On} \twoheadrightarrow N$ (This requires global choice in the kernel).

Theorem 4.9 is also a generalization of what are essentially some special simple cases worked out in [2, §1D] by Blass and Scedrov. Our proof is somewhat similar in spirit to theirs, using the M^* construction, but their proof did not make explicit use of forcing.

Proof of Theorem 4.9. Given N and M and $\mathbf{P}$ as in the hypotheses, assume without loss of generality that N is determined by $G = (\mathsf{Aut}(N)_{\mathbf{P}})^{M^+}$ as in Lemma 4.7(a & b).

We'll define Φ as a partial map $N_{\mathrm{ker}} \times \mathbf{P} \twoheadrightarrow N$. The map is

$$\Phi(z', p) = \begin{cases} x & \text{if } p \Vdash \dot{f}(\check{z}') = \check{x}. \\ \text{undefined} & \text{otherwise.} \end{cases}$$

It remains to check the definability of Φ.

The first step is to show the definability of

$$\mathsf{Dom}\,\Phi = \{\, \langle z', p\rangle \in N_{\mathrm{ker}} \times \mathbf{P} \mid (\exists x \in N)\, p \Vdash \dot{f}(\check{z}') = \check{x} \,\}.$$

Now $p \Vdash \dot{f}(\check{z}') = \check{x}$ if and only if for all generic $\Gamma \subset \mathbf{P}$ with $p \in \Gamma$, $f_\Gamma(z') = x$ (where $f_\Gamma := \mathsf{Val}_\Gamma\, \dot{f}$). By Lemma 4.7(c), we may replace "all generic Γ" in the previous sentence with "all generic Γ in M^+".

Write $p = \langle b', hb\rangle$. Using Lemma 4.7(d), we have $\{\, f_\Gamma(z') \mid p \in \Gamma \in M^+ \,\} = \{\, gj(z') \mid g \in G \text{ and } gj(b') = hb \,\} = \{\, gz \mid g \in hG_b \,\}$. By the previous paragraph, $\langle z', p\rangle \in \mathsf{Dom}\,\Phi$ iff this set is a singleton; i.e. $G_b \subseteq G_z$.

Let G^* be the group of automorphisms of A^* induced by G and j, as in the proof of Lemma 4.6. Clearly $G_b \subseteq G_z$ iff $G^*_{b'} \subseteq G^*_{z'}$. Let us write $b' \preccurlyeq z' :\leftrightarrow G^*_{b'} \subseteq G^*_{z'}$ Notice that this relation $\preccurlyeq$ on N^* is a definable class in N_{ker}. This is especially clear when $M^+ = M$, for in that case $G^* \in N_{\mathrm{ker}}$. In general, we have G, and hence G^*, almost in M, and since G^* is a pure set it follows that G^* is almost in $M_{\mathrm{ker}} = N_{\mathrm{ker}}$. So by Lemma 3.3, $\preccurlyeq$ is a definable (with parameters) subclass of N_{ker}.

Conclusion: $\mathsf{Dom}\,\Phi = \{\, \langle z', \langle b', d\rangle\rangle \mid \langle b', d\rangle \in \mathbf{P} \wedge z' \preccurlyeq b' \,\}$ is definable in N using pure parameters and $\mathbf{P}$. (Note that although we are using primed variables to stand for elements of N^*, we are not actually using the priming function j^{-1} in this definition.)

Now, suppose $p \Vdash \dot{f}(\check{z}') = \check{x}$, and let $\Gamma \subset \mathbf{P}$ with $p \in \Gamma$. We have

$$\begin{aligned} y \in x &\iff (\exists n' \mathrel{\mathsf{E}} z')\ f_\Gamma(n') = y \\ &\iff (\exists n' \mathrel{\mathsf{E}} z')(\exists q \leq p)\ q \Vdash \dot{f}(\check{n}') = \check{y} \\ &\iff (\exists n' \mathrel{\mathsf{E}} z')(\exists q \leq p)\ \langle n', q\rangle \in \mathsf{Dom}\,\Phi \ \wedge\ \Phi(n', q) = y. \end{aligned}$$

Thus, for $\langle z', p\rangle \in \mathsf{Dom}\,\Phi$, we have

$$\Phi(z', p) = \{\, \Phi(n', q) \mid n' \mathrel{\mathsf{E}} z' \wedge q \leq p \wedge \langle n', q\rangle \in \mathsf{Dom}\,\Phi \,\}.$$

This is a Δ definition of Φ by E-induction using only $\langle \mathbf{P}, \leq\rangle$, E, and $\mathsf{Dom}\,\Phi$ as parameters. Clearly E is definable with pure sets, and we have taken care of $\mathsf{Dom}\,\Phi$, so this finishes the proof. □

Proof of Main Theorem 4.1. The $\Rightarrow$ direction was proved in Lemma 4.5 and (for part (b)) Lemma 4.6. For the $\Leftarrow$ direction, suppose M is a generic extension of N. If M happens to be an extension of N by some almost homogeneous notion of forcing, then towards a proof of part (b), let $M^+ = M$. For the more general case of part (a), just let M^+ be some extension of M which is an almost homogeneous generic extension over both M and N; it follows from a result of Grigorieff [5, §4.9] that such M^+ always exists.

In M^+, let $G = \mathsf{Aut}(N)$, and let $\mathcal{F} = \{\, F < G \mid (\exists x \in N) G_x \subseteq F \,\}$. Let M_p be the (almost-)FMS submodel of M given by G and $\mathcal{F}$; the idea is to show that

$N = M_p$. Since N is transitive and consists of $(G, \mathcal{F})$-symmetric elements, easily we have $N \subseteq M_p$.

Let α be an ordinal large enough so that everything in M_p has a support $\{s\}$ with $s \in N_\alpha$, so that N_α is a normal base for M_p. Let $\mathbf{P}$ be the generator poset for M_p and M based on N_α and J, where $J = \{\, gj \mid g \in G \,\}$ for some isomorphism $j : N^* \to N$ in M. Since $G = \mathsf{Aut}(N)$ in M^+, J is the set of all isomorphisms $N^* \to N$ in M^+ and is therefore almost in N. Thus $\langle \mathbf{P}, \leq \rangle$ is almost in N. But the conditions of $\mathbf{P}$ are in $N^*_\alpha \times N_\alpha$, which is a subset of N, so by Lemma 3.3 $\langle \mathbf{P}, \leq \rangle$ is in N.

Now since $(M_p)_{\text{ker}} \cup \{\mathbf{P} \ \leq\} \subset N \subseteq M_p$, it follows from Theorem 4.9 that $N = M_p$. This completes the proof of Theorem 4.1. □

5 Concluding Remarks

We have the following corollary to the Main Theorem 4.1:

Corollary 5.1. *Let ψ_a be the sentence (in the language of set theory) "There is a notion of forcing $\mathbf{P}$ such that $\Vdash_{\mathbf{P}}$ (AC and all pure sets are in $\check{V}$)." Let ψ_b be the sentence "There is an almost homogeneous notion of forcing $\mathbf{P}$ such that $\Vdash_{\mathbf{P}}$ (AC and all pure sets are in $\check{V}$)."*

Then for any sentence ϕ in the language of set theory,

(a) $\text{ZFA} \vdash \psi_a \to \phi$ *iff* ϕ *holds in every almost-FMS model.*

(b) $\text{ZFA} \vdash \psi_b \to \phi$ *iff* ϕ *holds in every FMS model.*

Example 2.3 showed that an almost-FMS submodel of a given $M \models$ ZFAC is not necessarily an FMS submodel of M. However, it is an open question whether ψ_a and ψ_b from Corollary 5.1 are equivalent statements in ZFA. Indeed, the model in Example 2.3 is a model of ψ_b. Although it is not an FMS submodel of the given M, it is an FMS submodel of some other model of ZFAC.

Furthermore, not even the question of whether or not ψ_a or ψ_b are theorems of $\text{ZFA} + (\text{AC})^{\text{kernel}}$ has been resolved. To prove that they are not would essentially entail producing a model of $\text{ZFA} + (\text{AC})^{\text{kernel}}$ that is not an almost-FMS model.

References

[1] Andreas Blass. Injectivity, projectivity, and the axiom of choice. *Trans. Amer. Math. Soc.*, 255:31–59, 1979.

[2] Andreas Blass and Andre Scedrov. Freyd's models for the independence of the axiom of choice. *Mem. Amer. Math. Soc.*, 79(404):viii+134, 1989.

[3] Norbert Brunner. The Fraenkel-Mostowski method, revisited. *Notre Dame J. Formal Logic*, 31(1):64–75, 1990.

[4] Norbert Brunner and Jean E. Rubin. Permutation models and topological groups. *Rend. Sem. Mat. Univ. Padova*, 76:149–161, 1986.

[5] Serge Grigorieff. Intermediate submodels and generic extensions in set theory. *Ann. Math. (2)*, 101:447–490, 1975.

[6] Paul E. Howard. Limitations on the Fraenkel-Mostowski method of independence proofs. *J. Symbolic Logic*, 38:416–422, 1973.

[7] Thomas J. Jech. *The axiom of choice*. North-Holland Publishing Co., Amsterdam, 1973. Studies in Logic and the Foundations of Mathematics, Vol. 75.