1 Permutation Groups

Definition 1.1. A *permutation* of a set X is a bijection $X \to X$. The identity function $\mathrm{Id} : X \to X$ on X is an example of a permutation. If x and y are distinct elements of X, we may write $(x\,y)$ for the permutation that switches x and y and fixes all other elements of X.

A *group* of permutations of X is a non-empty set of permutations of X that is closed under composition and inverses. If G is a group of permutations, then a subset H of G which is also a group is called a *subgroup*, and we may write $H \leq G$.

For example, let G be the group of all permutations of X, and suppose there are at least two elements x and y in X. Then any group of permutations of X is a subgroups of G; examples include $\{\mathrm{Id}\}$ and $\{\mathrm{Id}, (x\,y)\}$ and G itself. $\{\mathrm{Id}, (x\,y), (x\,z)\}$ is not a subgroup (assuming z is a third distinct element of X), because it is not closed under composition.

Another example: Let $f_k : \mathbb{Z} \to \mathbb{Z}$ be defined by $f_k(x) = x + k$; then for any integer k, f_k is a permutation of $\mathbb{Z}$. The set $\{\, f_n \,:\, n \in \mathbb{Z} \,\}$ is a group of permutations. Note that $\{\, f_n \,:\, n \in \omega \,\}$ is not a group, though it is closed under composition.

Definition 1.2. If G is a group of permutations on X and $x \in X$, then the *stabilizer* of x in G is $\{\, \pi \in G \mid \pi(x) = x \,\}$, the set of all permutations in G that fix x. Write $\mathsf{sym}\, x$ or $\mathsf{sym}_G(x)$ or G_x for the stabilizer.

***Exercises* 1.3.**

- Show $G_x \leq G$.
- Show that Id is in every group of permutations.
- Show that if H and K are subgroups of G, then so is $H \cap K$.

2 $\in$-Automorphisms

Suppose V is the universe (or just a model of ZFA), and suppose the set A of atoms is countable in V. See the definition of $\wp^\alpha(S)$ on p. 45. Every $x \in V$ is in $\wp^\alpha(A)$ for some ordinal α.

Definition 2.1. The *rank* of $x \in V$ (notation: $\rho(x)$) is the least ordinal α such that $x \in \wp^\alpha(A)$.

Note:

- $\rho(x) = 0 \leftrightarrow x \in A$.
- $\rho(A) = \rho(0) = 1$.
- $\rho(\wp^\alpha(A)) = \rho(\alpha) = \alpha + 1$.
- For $x \notin A$, $\rho(x) = \sup\{\, \rho(y) \mid y \in x \,\} + 1$
  (so $\rho(x)$ is never a limit ordinal).

Let π be a permutation of A. We extend π to a function $V \to V$ using the following definition by transfinite induction on the rank of x:

$$\pi(0) = 0, \qquad \pi(x) = \{\, \pi(y) \,:\, y \in x \,\}.$$

***Exercises* 2.2.**

- Show by (transfinite) induction on α that for π as defined above, $\rho(x) = \alpha \leftrightarrow \rho(\pi x) = \alpha$.
- Prove that π as defined above is an automorphism of V.
- Prove that if x is in the kernel (i.e. x has no atoms in its transitive closure), then $\pi(x) = x$.

Remark: This extension of π defined above is the *unique* way to extend π to an automorphism of V. In other words, if any two automorphisms of V agree on A, then they must agree everywhere (i.e. they are equal).

The facts (a)-(h) on p. 46 follow easily from the definition or from the fact that π is an automorphism. Note especially (e) and (f).

3 Symmetric sets

In general, we start the construction of a permutation model by letting G be some group of permutations of A. For now, take G to be simply the group of all permutations of A. By the above process of extending permutations of A to automorphisms of V, we can think of G as the group of all automorphisms of V (a subgroup of the group of all permutations of V).

For any finite $E \subset A$, let

$$\mathsf{fix}_G(E) = \{\, \pi \in G \mid (\forall e \in E)\ \pi e = e \,\}.$$

Equivalently, we could write $\mathsf{fix}_G(E) = \{\, \pi \in G \mid \pi \restriction E = \mathrm{Id} \,\}$, or $\mathsf{fix}_G(E) = \bigcap_{e \in E} G_e$.

Definition 3.1. For now we are interested in the above notion for finite sets $E \subset A$, but the definition makes perfectly good sense for any subset $E \subset X$ when G is any group of permutations of X. $\mathsf{fix}_G(E)$ is sometimes called the *pointwise stabilizer of E (in G)*. Often, we just write $\mathsf{fix}(E)$ when G is clear from the context.

Note that the bigger E is, the smaller $\mathsf{fix}\, E$: We have

$$E \subset E' \quad \Rightarrow \quad \mathsf{fix}\, E \supset \mathsf{fix}\, E'$$
$$\mathsf{fix}(E \cup E') = \mathsf{fix}\, E \cap \mathsf{fix}\, E'$$

***Exercises* 3.2.**

- Show that $\mathsf{fix}\, E$ is a group.
- Let E be a finite subset of A with $|E| \geq 2$. Show $\mathsf{fix}\, E \subset \mathsf{sym}\, E$ and $\mathsf{fix}\, E \neq \mathsf{sym}\, E$.
- Let B be any subset of A. Show $G_B = G_{(A \smallsetminus B)}$.

Let $\mathcal{S} = \{\, E \subset A \mid E \text{ is finite} \,\}$.

Definition 3.3. We say that $x \in V$ is *symmetric (with respect to G and $\mathcal{S}$)* if there is an $E \in \mathcal{S}$ such that $\mathsf{fix}_G\, E \subseteq G_x$. If E is a subset of A such that $\mathsf{fix}_G\, E \subset G_x$, then we say E is a *support* for x, so x is symmetric if and only if x has a finite support.

***Exercise* 3.4.** Show that any finite or cofinite subset of A is symmetric (with respect to our G and $\mathcal{S}$).

Proposition 3.5. *Let $f : \omega \to A$ be a bijection. Then f is not symmetric.*

Proof. Suppose E is a finite subset of A. We will show that E cannot be a support for f by finding a permutation $\pi \in \mathsf{fix}\, E$ such that $\pi \notin G_f$.

Let a and b be distinct elements of $A \smallsetminus E$, and let $\pi = (a\, b)$. Let $n = f^{-1}a$.

It follows from $f(n) = a$ that $(\pi f)(\pi n) = \pi a$ (see facts (b) or (f) on p. 46, or consider that $\langle n, a\rangle \in f$ and think about what this means for πf using the inductive definition of π). Note $\pi n = n$, since n is in the kernel. Thus $(\pi f)(n) = b$. But $(\pi f)(n) \neq f(n)$ means $\pi f \neq f$, so $\pi \notin G_f$.

We have shown that no finite subset of A can be a support for f, and therefore f is not symmetric. □

***Exercises* 3.6.**

- Let R be a linear order on A. Show that R is not symmetric.
- Let B be an infinite, co-infinite subset of A. Show that B is not symmetric.
- Show that if $x \in V$ is symmetric, then πx is also symmetric.

Definition 3.7. A symmetric $x \in V$ is called *hereditarily symmetric* if every element of $\mathrm{TC}(x)$ is symmetric. Equivalently, x is hereditarily symmetric if and only if x is symmetric and all of its members are hereditarily symmetric.

***Exercises* 3.8.**

- Show that $\wp(A)$ is symmetric but not hereditarily symmetric.
- Let $\wp(A)^{HS} = \{\, x \subseteq A \mid x \text{ is symmetric} \,\}$. Show that $\wp(A)^{HS}$ is hereditarily symmetric.

Later, we will prove

Theorem 3.9. *The class of hereditarily symmetric sets is a model of ZFA.*

Assuming this theorem, we can conclude

Theorem 3.10. *If* ZFA *is consistent, then so is* ZFA $+ \neg$AC. *In fact, if* ZFA *is consistent, then the theory* ZFA + "There is an infinite set with no partition into two infinite sets" *is consistent, and therefore it cannot be proved from the axioms of* ZFA *alone that every infinite set has a partition into two infinite sets.*

Proof. The assumption that ZFA is consistent allows us to assume that we have a model V of ZFA, and we define the class HS of hereditarily symmetric elements of V as in this section. By Theorem 3.9, the class HS is a model of ZFA. However, HS has no partition P of A into two infinite sets: Such a P would have an infinite, co-infinite subset $S \subset A$ as a member, and no such S is symmetric by the second part of Exercise 3.6. □

4 Symmetry With Finite Supports

We will now generalize the idea of "symmetric" given in the last section. Notice that Definition 3.3 actually said what it meant for $x \in V$ to be symmetric " *with respect to G and $\mathcal{S}$*". Thus, we obtain new notions of "symmetric" in two ways: (1) By replacing the group of all permutations of A used in the previous section with some smaller group G, and (2) by replacing $\mathcal{S}$ with some other family.

In this section we consider only the first method. Typically, the groups G that we use are described as $\{$ all automorphisms π of V such that $\pi x = x$ $\}$, for some set x, but other groups are also used.

Notation. Let $\mathsf{Sym}(A)$ be the group of all permutations of A (note the capitol letter "S"; this is a more specific notion than the sym of 1.2). By extending these permutations, we know we can think of $\mathsf{Sym}(A)$ as the group of all automorphisms of V, and we may call this group $\mathsf{Aut}(V)$ as well as $\mathsf{Sym}(A)$.

For $x \in V$, let $\mathsf{Sym}(x)$ be stabilizer of x in $\mathsf{Sym}(A)$.

Let's consider a specific example: Let $\prec$ be a dense linear order on A, and let G be the group of permutations of A that respect this ordering. In other words, $G = \mathsf{Sym}(\prec)$. By rereading Definition 3.3, but with this new G in mind, we arrive at a modified notion of "symmetric". (For now, we continue to base our notion of symmetry on finite supports.)

The next lemma points out some of the permutations that are available to us in this G.

Lemma 4.1. *Let $\prec$ be any dense linear ordering on A, and let a and b be distinct elements of A with $a \prec b$. As usual, we write (a, b) for the open interval, i.e. the set of elements of A strictly between a and b. Then there is an order preserving permutation $\pi : A \to A$ such that π moves all elements in (a, b) and fixes all other elements of A.*

Partial proof. We give a proof in the case where A is countable. (It is true in general if A is well-orderable, but may not always be true if the axiom of choice is not assumed.) It is a well-known result of Cantor that every countable dense linear ordering is order-isomorphic to $\langle \mathbb{Q}, < \rangle$ (the set of rational numbers with the usual ordering). Thus, since (a, b) is countable and the ordering on (a, b) is dense, we may put rational indices on the elements of (a, b) in such a way that $(a, b) = \{ a_q \mid q \in \mathbb{Q} \}$ and $a_q \prec a_r \leftrightarrow q < r$. Now define $\pi(a_q) = a_{q+1}$, and $\pi(c) = c$ when $c \in A$ is not an element of (a, b). $\square$

***Exercises* 4.2.**

- Show that if $f : \omega \to A$ is one-to-one, then f is not symmetric. (Here we mean symmetric with respect to $G = \mathsf{Sym}(\prec)$.)
- Using the new notion of "symmetric" gives us a new notion of "hereditarily symmetric" in the obvious way. Assume that Theorem 3.9 holds for this new notion of hereditarily symmetric, and assume that ZFA is consistent. Show that ZFA + "not every linearly ordered set can be well-ordered" is consistent.

In general, if you want to make a model of ZFA in which A has structure R (like a relation) but has as little structure beyond that as possible, a good way is often to let $G = \mathsf{Sym}(R)$ and

define symmetry accordingly (using finite supports). For example, if you want a hereditarily symmetric partition P of A into two infinite sets but no symmetric linear order on A, then $G = \mathsf{Sym}(P)$ will do the job.

***Exercises* 4.3.**

- Find a G such that some partial order of A with no maximal element is be symmetric with respect to G, but such that no linear order on any infinite subset is symmetric.
- Find a G such that there is a hereditarily G-symmetric family P of disjoint 2-element sets, but such that no choice function on P is symmetric.
- Let $\prec$ be a linear order of A isomorphic to the usual ordering of $\mathbb{Z}$. Show that every $x \in V$ is symmetric with respect to $G = \mathsf{Sym}(\prec)$, so this G turns out to be not very useful for our purposes.

Remark. Sometimes changing the group doesn't end up changing the notion of symmetry! For example, let F be the set of all permutations of A that move only finitely many elements of A. It turns out that symmetry with respect to F is the same as symmetry with respect to $\mathsf{Sym}(A)$ (the symmetry of the previous section). (We won't prove this here, but to get a little feel for what is happening, look back at your solution to the second part of Exercise 3.6. It *probably* works without any change even for F-symmetry.)

5 More General Families of Supports

So far, we have been calling a set x symmetric iff it has a support that is a finite set of atoms. Often, it is useful to allow more sets to be considered symmetric. We choose a family B satisfying the three conditions in the following definition—we'll call B a "base of supports"—and say that a set is symmetric iff it has a support in B. Again, Theorem 3.9 will hold for this more general notion of symmetry.

Definition 5.1. Fix a group $G < \mathsf{Aut}(V)$. For any sets E and x, we say that E *supports* x if $\mathsf{fix}_G(E) < G_x$. A *base of supports (for G)* is a set B such that:

(1) $G_B = G$.

(2) Every atom is supported by some $b \in B$.

(3) For each b_0 and b_1 in B, there is a c in B such that c supports every element of $b_0 \cup b_1$.

***Exercise* 5.2.** For any $G < \mathsf{Aut}(V)$, show that the set of all finite subsets of atoms is a base of supports.

The family of finite subsets of A is an example of what is called a "normal ideal"; different normal ideals are often used as bases of supports.

Definition 5.3. A *(G-)normal ideal on A* is a set $I \subset \wp(A)$ such that

(1) Every subset of any member of I is in I.

(2) The union of any two members of I is in I.

(3) $\{a\} \in I$ for each $a \in A$.

(4) $G_I = G$.

(The concept of "ideal" is common in many other contexts besides constructing permutation models. Given any set X, an *ideal on* X is a non-empty family $I \subset \wp(X)$ satisfying conditions (1) and (2) above.)

***Exercises* 5.4.**

- Let κ be any infinite cardinal. Let $I = \{\, i \subset A \mid |i| < \kappa \,\}$. Show that I is a normal ideal (in fact, G-normal for any G). Do you need to use the axiom of choice?
- Show that any normal ideal is a base of supports. (And by this I mean: Show that any G-normal ideal is a base of supports for G.)
- Let $\prec$ be a dense linear order on A, and let $G = \mathsf{Sym}(\prec)$. let B be the set of subsets of A that are bounded with respect to $\prec$. Show that B is a base of supports for G.
- Suppose A is uncountable, and let B be the set of countable partitions of A. Show that B is a base of supports (for any G).

Let CAC (short for "countable axiom of choice") be the principle "every countable family of non-empty sets has a choice function". One way to get a permutation model in which CAC holds (but AC does not hold) is to start with uncountably many atoms, and then use the ideal of countable supports.

Theorem 5.5. *Let A be uncountable, let $G = \mathsf{Sym}(A)$, and say that x is symmetric iff it is supported by some countable set of axioms. Suppose $C = \{\, c_i \mid i \in \omega \,\}$ is some hereditarily symmetric countable family of non-empty sets. Then every choice function f for C is symmetric.*

Proof. Let f be a choice function for C. For each $i \in \omega$, we have $f(c_i) \in c_i \in C$. Since C is assumed to be hereditarily symmetric, we know that also both c_i and $f(c_i)$ are symmetric. Let s_i be a countable support for c_i, and let t_i be a countable support for $f(c_i)$. Now let $S = \bigcup_{i\in\omega}(s_i \cup t_i)$. Observe that S is a support for f. Since S is also countable subset of A, we have that f is symmetric. □

Now assuming Theorem 3.9 holds for the notion of symmetry used in the previous theorem, we have

Corollary 5.6. *There is a model of ZFA in which CAC holds but AC does not hold.*

Proof. Take HS to be the class of hereditarily symmetric sets, where symmetry is defined with countable supports as in the previous theorem. That theorem shows that every countable family of non-empty sets $C \in HS$ has a symmetric choice function f. Now this f is symmetric, and all the elements of its domain and range are hereditarily symmetric; it follow that f is hereditarily symmetric. So the choice function f is in HS, and that means CAC holds in HS.

On the other hand, if A is uncountable, then there is no symmetric well-ordering of A. So AC does not hold in HS. □

6 The Main Theorem

It is time now to state the *real* definitions of symmetry (this will subsume Definition 3.3) and of permutation model, and to work on proving that permutation models are in fact models of ZFA (general version of Theorem 3.9).

Definition 6.1. Let A be the set of atoms in the universe, V. (Assume A is infinite; this definition makes sense in any case but isn't very useful if A is finite.) Let G be a subgroup of $\mathsf{Sym}(A)$ (or, equivalently, of $\mathsf{Aut}(V)$), and let B be a base of supports for G.

We say that $x \in V$ is *symmetric (with respect to G and B)* if there is an $E \in B$ that supports x (i.e., $\mathsf{fix}_G\, E \subseteq G_x$). As before, a symmetric $x \in V$ is called *hereditarily symmetric* iff every element of $\mathrm{TC}(x)$ is symmetric.

Let $\mathcal{N}_{G,B}$ be the class of sets that are hereditarily symmetric with respect to G and B. We may also simply write $\mathcal{N}_G$ or $\mathcal{N}_B$ or $\mathcal{N}$ when the group and/or base of supports are clear from the context. We call $\mathcal{N}_{G,B}$ the *permutation model* obtained from G and B.

Remark. The notation V is typically used to stand for the "real" universe of all sets (in some platonistic sense), without any atoms. To build a useful permutation model, we need infinitely many atoms. Furthermore, for some purposes we find having uncountably many atoms to be convenient, while for others we only want countably many. Instead of assuming at different times that the set of atoms "in V" has different cardinalities, it may be preferable to leave V alone and say something like "let $\mathcal{M}$ be a model of ZFAC with countably many atoms". Then to make a permutation model, we let take some G and B in $\mathcal{M}$, and define the collection of symmetric elements of $\mathcal{M}$, as in the above definition but with $\mathcal{M}$ in place of V. The resulting $\mathcal{N}_{G,B}$ is called the *permutation submodel of $\mathcal{M}$ obtained from G and B.*

We will eventually justify calling $\mathcal{N}_{G,B}$ a permutation "model" by proving that it is a model of ZFA. For now, just think of a permutation model as a certain class of sets.

***Exercises* 6.2.** Let $\mathcal{N} = \mathcal{N}_{G,B}$ be a permutation model.

- Show that if x and y are in $\mathcal{N}$, so is the ordered pair $\langle x, y\rangle$.
- Show that $G_{\mathcal{N}} = G$.
- Let $X \in \mathcal{N}$, and define $\wp(X)^{\mathcal{N}} := \{\, s \subseteq X \mid s \text{ is symmetric} \,\}$. Show that $\wp(X)^{\mathcal{N}} \in \mathcal{N}$.

Definition 6.3. Let ϕ be a formula in the language of set theory, and let X be some set, or some definable class, or some variable not occurring in ϕ. The *relativization of ϕ to X*, denoted ϕ^X, is the formula obtained from ϕ by restricting each quantifier to X. In other words, we replace each instance of $\exists v$ in ϕ with $(\exists v \in X)$; likewise, we we replace each instance of $\forall v$ in ϕ with $(\forall v \in X)$.

Let's look at a fairly simple formula as an example. Take η to be the Axiom of Extensionality:

$$\forall x \forall y([\forall z(z \in x \leftrightarrow z \in y)] \to x = y).$$

Then, for example, η^3 is

$$(\forall x \in 3)(\forall y \in 3)([\forall z \in 3(z \in x \leftrightarrow z \in y)] \to x = y).$$

You might check that η^3 happens to be a true statement, whereas if $X = \{\,\{0,2\},\{0,3\}\,\}$, then η^X is false.

As a more useful example, let ϕ be the sentence "A is well-orderable" (of course, ϕ is really a more formal version in the language of set theory). A fundamental point of this whole pamphlet is that if $\mathcal{N}$ is a non-trivial permutation model, then $\phi^{\mathcal{N}}$ is false.

***Exercises* 6.4.**

- Let x and y be variables, and let ϕ be a formula. Show that $(\phi^x)^y$ is just $\phi^{(x \cap y)}$. (It may help to remember that $(\exists z \in x)\psi$ is itself just an abbreviation for $\exists z(z \in x \wedge \psi)$, and similarly for $\forall$.)
- Show that if x is transitive, then η^x holds (where η is Extensionality).
- Show that if x is any set or class, then (Axoim of Foundation)^x holds.
- Find a list of axioms for ZF somewhere. For which axioms ϕ does ϕ^{V_ω} hold, and for which axioms is ϕ^{V_ω} false?
- Same as above, but for $\phi^{V_{\omega+\omega}}$.

Definition 6.5. Let $\mathcal{M}$ be a set or a class. If $\phi^{\mathcal{M}}$ holds, then we say $\mathcal{M}$ is an *$\in$-model*, or just a *model* of ϕ. Likewise, if T a set of sentences in the language of set theory, then an *($\in$-)model* of T is just a model of every sentence in T.

Let Aω denote the Axiom of Infinity. In one of the exercises above, you should have found that V_ω is a model of the set of sentences (ZFC $-$ Aω) $+ \neg$Aω. This shows that that set of sentences is consistent, and hence we cannot prove Aω from the other axioms of ZFC.

Our aim is to create models of ZFA $+ \neg$AC; this will show that AC is not provable from the axioms of ZFA. Of course, that conclusion is neglecting the possibility that ZFA is simply inconsistent. We are relying on the following theorem, whose proof we shall omit.

Theorem 6.6. *Let S be a set of sentences (think ZF or ZFA... something that we assume is consistent), and let T be another set of sentences (whose consistency we are interested in, like* ZFA $+ \neg$*AC). If, working in S, one can find a model of T, then T is consistent if S is.*

You, the reader, are almost ready to prove the main theorem. A couple of lemmas will be useful. First, a lemma that has use beyond the study of permutation models.

Lemma 6.7. *Let $\mathcal{M}$ be a transitive class such that for all ordinals α, the set $\mathcal{M} \cap V_\alpha$ is a member of $\mathcal{M}$. (Generally, when $\mathcal{M} \cap V_\alpha \in \mathcal{M}$, we shall use the notation $\mathcal{M}_\alpha := \mathcal{M} \cap V_\alpha$.) If $\mathcal{M}$ is a model of all axioms of ZF except perhaps those of Replacement and Powerset, then in fact $\mathcal{M}$ is a model of ZF.*

Remark. The lemma is still true replacing ZF with ZFA or ZFC or ZFAC, by straightforward modifications of the proof.

***Exercises* 6.8.**

- Prove that an $\mathcal{M}$ satisfying the hypotheses of Lemma 6.7 contains all ordinals. (You don't need to use any of the stuff about $\mathcal{M}$ satisfying lots of ZF axioms.)
- Prove Lemma 6.7.

Next, a very specific (and straightforward) lemma for the proof of the main theorem.

Lemma 6.9. *Suppose $y_1, \ldots, y_n$ are all members of the permutation model $\mathcal{N}_{G,B}$, and $\{y_1, \ldots, y_n\}$ supports x. Then x is symmetric with respect to G and B.*

***Exercise* 6.10.** Prove Lemma 6.9.

Theorem 6.11 (The Main Theorem). *Each permutation model $\mathcal{N} = \mathcal{N}_{G,B}$ is a model of ZFA.*

Proof for the Union Axiom. Let $x \in \mathcal{N}$. We need to show:

$$(\exists U \in \mathcal{N})\,(\forall Y \in \mathcal{N})(\forall z \in \mathcal{N})\,[z \in Y \ \wedge\ Y \in x \ \rightarrow z \in U].$$

Remark: The formula above spells out the relativization of the Union Axiom to $\mathcal{N}$. However, note that the conditions "$z \in Y \wedge Y \in x$" in the formula above make some of the "$\in \mathcal{N}$"s redundant, and so to show

$$(\exists U \in \mathcal{N})\,\forall Y\,\forall z\,[z \in Y \wedge Y \in x \ \rightarrow z \in U].$$

is equivalent.

It is clear that taking $U = \bigcup x$ will satisfy the formula, as long as we can show that $\bigcup x$ is a member of $\mathcal{N}$. We can see that $\bigcup x$ is a subset of $\mathcal{N}$, since $\bigcup x \subseteq \mathrm{TC}(x) \subset \mathcal{N}$ (using the fact that $\mathcal{N}$ is transitive).

Now observe that $\{x\}$ is a support for $\bigcup x$, since if $g \in \mathsf{fix}_G(x)$ then

$$g\bigcup x \ = \ \bigcup gx \ = \ \bigcup x,$$

where the first equality holds because g is an automorphism. Thus x is symmetric by Lemma 6.9.

So $x \in \mathcal{N}$, and this completes the verification of (Union Axiom)$^{\mathcal{N}}$. $\square$

***Exercises* 6.12.** Complete the proof of Theorem 6.11 as follows.

- Show that $\mathcal{N}$ is a model of Extensionality and Foundation and Infinity.
- Show that $\mathcal{N}$ is a model of the Pairing Axiom.
- Suppose ϕ is some formula and x and $y_1, \ldots, y_n$ are some sets and $s = \{\, z \in x \mid \phi^{\mathcal{N}}(z, x, y_1, \ldots, y_n)\,\}$. (By the way: How do I know there is such a set?) Show that $\{x, y_1, \ldots, y_n\}$ supports s. (Hint: One of the Exercises in 6.2 should be helpful.)
- Show that $\mathcal{N}$ is a model of the Comprehension schema.
- Show that $\mathcal{N}$ is a model of the Power Set axiom and Replacement schema. (Hint: Of course, Lemma 6.7 might help. Again, looking at part of Exercises 6.2 should be helpful.)