Group-oriented (t, n) threshold digital signature scheme and digital multisignature

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Abstract: The paper presents group-oriented (t, n) threshold digital signature schemes based on the difficulty of solving the discrete logarithm problem. By employing these schemes, any t out of n users in a group can represent this group to sign the group signature. The size of the group signature and the verification time of the group signature are equivalent to that of an individual digital signature. In other words, the (t, n) threshold signature scheme has the following five properties: (i) any group signature is mutually generated by at least t group members; (ii) the size of the group signature is equivalent to the size of an individual signature; (iii) the signature verification process is simplified because there is only one group public key required; (iv) the group signature can be verified by any outsider; and (v) the group holds the responsibility to the signed message. In addition to the above properties, two of the schemes proposed do not require the assistance of a mutually trusted party. Each member selects its own secret key and the group public key is determined by all group members. Each group member signs a message separately and sends the individual signature to a designated clerk. The clerk validates each individual signature and then combines all individual signatures into a group signature. The (n, n) threshold signature scheme can be easily extended to become a digital multisignature scheme.

1 Introduction

The threshold cryptosystem was first introduced by Desmedt [4] in 1987. In this system, each group, instead of each group member, publishes a single group public key. An outsider can use this single public key to send an encrypt message to this group. The received ciphertext can only be deciphered properly when the number of participating group members is larger than or equal to the threshold value. All up-to-date solutions for the group-oriented threshold cryptosystem can be classified into the following two categories: (i) solutions with the assistance of a mutually trusted party to decide the group secret key and generate individual secrets for all group members [5, 8, 10]; and (ii) solutions without the assistance of a mutually trusted party [13, 16]. As pointed out by Ingemarsson and Simmons [11], in most applications a trusted party in a group does not exist. This situation becomes more common in some commercial and/or international applications. Thus, the solutions without the assistance of a mutually trusted party become very attractive.

The threshold signature scheme is very similar to the threshold cryptosystem. In a threshold signature scheme, the group signature can only be generated when the number of participating group members is larger than or equal to the threshold value. Any outsider can use a group public key to verify this group signature. Boyd [2] proposed the first (n, n) group-oriented signature based on the RSA assumption [18] in 1986. In his scheme, if the number of group members is larger than two, most of the members can only sign the message blindly. Chaum and van Heyst [3] proposed another (n, n) group-oriented signature scheme in Eurocrypt '91. In their scheme, the number of listed group public key is not limited to one. In 1991, Desmedt and Frankel [6] proposed the first (t, n) threshold digital signature scheme based on the RSA assumption. In their scheme, a trusted key authentication center (KAC) is required for determining the group secret key and all members' secret keys. A group-oriented (n, n) undeniable signature scheme [9] was presented in Auscrypt '92. Unlike the normal signature that can be verified by any outsider, the undeniable signature can only be verified with the cooperation of all signers.

The threshold signature scheme can be applied to solve the problem of issuing checks for a corporation. For security reasons, it may be a company's policy that checks be signed by at least t individuals rather than one person. More formally, a (t, n) threshold digital signature scheme is designed to break the group secret key K into n different 'shadows', $K_1, K_2, \ldots, K_n$, so that:

(i) with knowledge of any t 'shadow' ($t < n$), the group signature can be easily produced;

(ii) with knowledge of any $t - 1$ or fewer 'shadows', it is impossible to forge a group signature;

(iii) it is impossible to derive the group secret key from the released group signature and all partial signatures; and

(iv) it is impossible to derive any secret 'shadow' from the released group signature and all partial signatures.

This definition is very similar to the definition of a (t, n) threshold secret sharing scheme. The major difference is that, in the secret sharing scheme, since the secret 'shadows' are exchanged among users and the group secret key is derived after each secret reconstruction process, the group secret key can only be used once if no other encryption scheme has been used; however, in the signature scheme, since the secret 'shadows' and the

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group secret key are never revealed in the cleartext form, the group secret key can be used repeatedly. In other words, the (t, n) threshold signature scheme integrates the secret sharing scheme and the digital signature scheme together to provide an efficient solution for group-oriented application.

The group signature is a kind of digital multisignature which is generated by multiple signers with knowledge of multiple secrets. Generally speaking, one of the major differences between a hand-written and a digital multisignature is the size of the multisignature. In a hand-written multisignature the size is linear in the number of signers but, in a digital multisignature, the size can be identical to a single signature. Digital multisignature is just a string of binary bits that can only be generated with the knowledge of a set of secret keys. An outsider can easily verify the authenticity of a given message based on the multisignature and the signers' public keys. In other words, digital multisignature is just a one-way trapdoor function. With the knowledge of a set of the trapdoor secrets, it is possible to generate a one-way output as the digital multisignature. Thus, it is not necessary for the size of the digital multisignature to be linear in the number of signers.

2 Modified ElGamal signature scheme

This scheme was developed from ElGamal's original signature scheme [7] in 1985, the modified ElGamal scheme being proposed by Agnew *et al.* [1] in 1990.

The scheme starts with a large prime, p, and a primitive element, α, of GF(p), which are publicly known. In order to provide adequate security, Pohlig and Hellman [17] indicate that p should be selected such that $p-1$ contains at least one large prime factor. They recommend choosing $p = 2p' + 1$, where p' is also a large prime. A one-way function f also needs to be made public.

In this scheme, each user selects a random exponent z from GF(p) as his private key. Suppose user A randomly selects a number, z_A, from $[1, p-1]$. Then A computes

$$y_A = \alpha^{z_A} \bmod p$$

as A's public key. Assume that A wants to sign a message m. User A then randomly selects a number k from $[1, p-1]$ and computes

$$r = \alpha^k \bmod p$$

User A now solves the congruence

$$z_A m' = kr + s \bmod p - 1$$

or

$$s = z_A m' - kr \bmod p - 1 \quad (1)$$

for integer s, where $0 \leqslant s \leqslant p-2$ and $m' = f(m)$. The one-way function f is used to increase the redundancy of m to avoid ElGamal's attack [7]. The signature for message m is then the ordered pair $\{r, s\}$.

Upon receiving the set of $\{m, r, s\}$, any user can verify the signature of message m as

$$y_A^{m'} = r^r \alpha^s \bmod p, \quad (2)$$

where $m' = f(m)$. There are two reasons for building a scheme based on this modified scheme.

(i) In order to simplify the signature verification, it is desirable to use a universal modulus p for all members to sign their individual signatures. In the RSA scheme, however, if the modulus n is universal and each member needs to know the factoring of n in order to decide his secret key, there will be no secret among all internal members. In this modified scheme, the modulus p contains no secret information at all.

(ii) As described later, the multiple individual signatures, $\{r_i, s_i\}$, $i = 1, 2, \ldots, n$, which corresponds to the same message, produced by this scheme, can be combined into a multisignature without any data expansion. In addition, the multisignature can also be verified very efficiently. However, the original ElGamal scheme and the modified scheme proposed by Agnew *et al.* cannot combined multiple individual signatures efficiently.

2.1 Security discussion

The security analysis of the modified scheme is very similar to the security analysis of that proposed by Agnew *et al.* [1]. Here, some possible attacks are briefly examined.

(i) An attacker might try to solve the secret key, z_A, based on the linear eqn. 1. For a given message and a signature pair, eqn. 1 involves two unknown parameters, z_A and k. For any increment number of the message and the corresponding signature pair, the unknown parameter is also increased by one. Therefore, the number of unknown parameters is always larger than the number of available equations. This attack cannot work successfully.

(ii) The attacker might try to forge a signature pair of a given message based on eqn. 2. He might try to randomly select an integer r' first and then compute the corresponding s' based on eqn. 2. Obviously, this difficulty is equivalent to solving the discrete logarithm problem. On the other hand, he might try to randomly select an integer s' first and then compute the corresponding r'. This is an extremely difficult problem, and in all likelihood, is more difficult than the discrete logarithm problem itself [1].

3 (n, n) Threshold signature scheme without the assistance of a mutually trusted party and digital multisignature scheme

Assume that the group policy requires that a group signature must be mutually signed by all group members. A group-oriented signature scheme consists of three phases: the public keys generating phase, the group signature generating phase, and the group signature verification phase. During the group and member public keys generating phase, all members select their individual secret keys and work together to determine the group public key. In the group signature generating phase, each group member receives a copy of the message to be signed. A member than signs the message and sends it along with the signature to a designated clerk. The designated clerk is responsible for collecting and authenticating each individual signature signed by each member, and produces a combined group signature. There is no secret information associated with the designated clerk.

3.1 Public keys generating phase

The scheme allows each member in a group to select his own secret key and all members to determine the grup public key together. Assume that there are n group members.

A large prime p, a primitive element α, of GF(p), and a one-way function f need to be made public.

Each member randomly selects an integer z_i from

$[1, p-1]$ and computes a corresponding public key as

$$y_i = \alpha^{z_i} \bmod p$$

The group public key y is then determined by all members as

$$y = \prod_{i=1}^{n} y_i \bmod p$$

3.2 Group signature generating phase

The scheme allows group members to sign a message simultaneously. This phase can be further divided into two parts.

Part 1: Generating and verifying individual signature. The procedure for generating an individual signature can be described as follows.

(i) Each member u_i randomly selects a number k_i from $[1, p-1]$ and computes

$$r_i = \alpha^{k_i} \bmod p$$

(ii) The result $\{r_i\}$ is broadcasted to all members. Once r_i, $i = 1, 2, \ldots, n$, from all members are available through the broadcast channel, each member computes the value r as

$$r = \prod_{i=1}^{n} r_i \bmod p$$

(iii) Member u_i uses his secret keys z_i and k_i, to sign the message m based on the modified signature scheme and solves the equation

$$s_i = z_i m' - k_i r \bmod p - 1$$

for integer s_i, where $0 \leqslant s_i \leqslant p-2$ and $m' = f(m)$, and transmits $\{m, s_i\}$ to the clerk. Note here that the individual signature, $\{r_i, s_i\}$, is a partial signature of message m.

Once the clerk receives the individual signature $\{r_i, s_i\}$ from u_i, he needs to verify the validity of this signature. To do this the clerk uses u_i's public key y_i to compute

$$y_i^{m'} = r_i^r \alpha^{s_i} \bmod p$$

where $m' = f(m)$. If the equation holds true, the partial signature $\{r_i, s_i\}$ of message m received from u_i has been verified.

Part 2: Generating the group signature. Once all partial group signatures are received and verified by the clerk, the group signature of message m can be generated as $\{r, s\}$, where $s = s_1 + s_2 + \cdots + s_n \bmod p - 1$.

3.3 Group signature verification phase

After receiving the group signature $\{r, s\}$, of the message m, an outsider needs to use the group public key y to verify the validity of the signature. The verification procedure is given as

$$y^{m'} = r^r \alpha^s \bmod p$$

where $m' = f(m)$. If the equation holds true, the group signature $\{r, s\}$ has been verified.

Theorem: If $y^{m'} = r^r \alpha^s \bmod p$, the group signature $\{r, s\}$ has been verified.

Proof: With the knowledge of secret key z_i, user u_i is able to generate its partial signature $\{r_i, s_i\}$ for message m to satisfy

$$y_i^{m'} = r_i^r \alpha^{s_i} \bmod p$$

where $m' = f(m)$. Multiplying the above equation for $i = 1, 2, \ldots, n$ yields

$$\prod_{i=1}^{n} y_i^{m'} = \prod_{i=1}^{n} r_i^r \alpha^{s_i} \bmod p$$

This relation is the same as

$$\prod_{i=1}^{n} (y_i)^{m'} = \left(\prod_{i=1}^{n} r_i^r \right) (\alpha)^{\sum_{i=1}^{n} s_i} \bmod p$$

Since

$$r = \prod_{i=1}^{n} r_i \bmod p$$

$$s = s_1 + s_2 + \cdots + s_n \bmod p - 1$$

and

$$y = \prod_{i=1}^{n} y_i \bmod p$$

then

$$y^{m'} = r^r \alpha^s \bmod p$$

3.4 Security

The security analysis of this signature scheme is very similar to the security analysis of the modified signature scheme just described. Here, some possible attacks are briefly examined.

(i) Instead of satisfying $y_i^{m'} = r_i^{r_i} \alpha^{s_i} \bmod p$ as in the modified signature scheme, the partial signature in the group signature scheme needs to satisfy $y_i^{m'} = r_i^r \alpha^{s_i} \bmod p$. Since r_i and r are public values and contain no secrets, the attacker cannot reveal any secret from this equation.

(ii) With the knowledge of all partial signatures and the group signature, the attacker needs to solve the equation

$$\begin{aligned}(z_1 + z_2 + \cdots + z_n)m' &= (k_1 + k_2 + \cdots + k_n)r \\ &\quad + (s_1 + s_2 + \cdots + s_n) \bmod p - 1\end{aligned}$$

in order to determine the secret keys. It has the same difficulty as in the modified signature scheme.

(iii) An attacker might try to impersonate user u_i by randomly selecting a r_i' and then obtaining

$$r' = \left(\prod_{j=1, j \neq i}^{n} r_j \right) r_i' \bmod p$$

The attacker needs to find a value s_i' to satisfy the equation as $y_i^{m'} = r_i'^{r'} \alpha^{s_i} \bmod p$. This difficulty is equivalent to solving the discrete logarithm problem. On the other hand, an attacker might first try to randomly select a pair of (r', s_i'), then broadcast a forged r_i', to satisfy

$$r' = \left(\prod_{j=1, j \neq i}^{n} r_j \right) r_i' \bmod p$$

Since $y_i^{m'} \neq r_i'^{r'} \alpha^{s_i} \bmod p$, this forged partial signature (r_i', s_i') cannot satisfy the signature verification equation. It is therefore concluded that, although one of the partial signatures r_i from each member is not authenticated by other members and the attacker can easily change this value, to place a successful attack is infeasible. On the other hand, if it is necessary, each member can still sign this partial signature and then make the signature of r_i and r_i itself available on the broadcast channel.

(iv) If the clerk is allowed to collect all r_i from the members and to broadcast the productive result r for all

members to sign accordingly, there will be a possible active attack associated with the clerk. This is because, instead of broadcasting r, the clerk broadcasts $r' = r^t \bmod p$ for all members to use to sign their signatures. With the knowledge of the signature pair (r, s) for the message m', the clerk can successfully forge a signature pair (r', st) for the message m'', where $m'' = m't \bmod p - 1$. Since t is a random integer, this attack can be applied to forge any message. Thus, it is recommended that all group members to compute their own r to avoid this attack.

3.5 Other features

(i) In this scheme, a signature signed only by partial members cannot be verified correctly by an outsider. In other words, a valid group signature must be mutually generated by all members.

(ii) The group signature in this scheme consists of a pair of $\{r, s\}$. The n individual signatures produced by all members consist of n pairs of $\{r_i, s_i\}$. Thus, the scheme combines n individual signatures into a single signature.

(iii) The group signature verification process requires two modular exponentiations. However, the varification process for n individual signatures requires $2n$ modular exponentiations. Thus, this scheme speeds up the verification process by a factor of n.

(iv) The same scheme can be easily applied to solve the digital multisignature problem. Instead of combining n individual signatures in a group-oriented signature, the digital multisignature scheme should be able to combine any number of individual signatures into a multisignature. Also, instead of using a fixed group public key to verify the signature in the group-oriented signature scheme, the verifier in the digital multisignature scheme should use all signer's public keys to verify the multisignature. There are two properties that need to be achieved in the design of an optimal digital multisignature scheme: (a) the size of the multisignature should be equivalent to the size of an individual's signature; and (b) the verification process of multisignature should be almost equivalent to the verification process of an individual's signatures. References 13, 15 and 16 provide for more information on this topic. All existing digital multisignature schemes are based on the factoring problem. Since each user selects a different modulus n for their public key, there are two problems associated with this approach: (a) the signing order has certain restrictions (i.e. the moduli associated with signers should be arranged in an ascending order); and (b) the multisignature verification process requires all different moduli n (i.e. the required operation is linear in the number of signers). Thus, they are not the optimal digital multisignature schemes. The proposed multisignature scheme is the first scheme based on the discrete logarithm problem. Since all users use the same modulus p in this scheme, it allows users to sign the same message simultaneously. In addition, it can compress n partial signatures into a multisignature without any data expansion and it also simplifies the verification process significantly. Thus, the proposed scheme is optimal.

4 (t, n) Threshold digital signature scheme with the assistance of a mutually trusted party

This scheme utilises the cryptographic techniques of Shamir's perfect secret sharing scheme [19] based on the Lagrange interpolating polynomial and the digital signature algorithm proposed by NIST [20]. The trusted key authentication center (KAC) is responsible for selecting all parameters, the group secret key and all secret shadows for group members. The KAC selects:

(i) p, a large prime modulus, where $2^{511} < p < 2^{512}$,

(ii) q, a prime divisor of $p - 1$, where $2^{159} < q < 2^{160}$,

(iii) $\{a_i, \text{ for } i = 0, \ldots, t-1\}$, and $f(x) = a_0 + a_1 x + \cdots + a_{t-1}x^{t-1} \bmod q$, each a_i is a random integer with $0 < a_i < q$,

(iv) α, where $\alpha = h^{(p-1)/q} \bmod p$, h is a random integer with $1 \leqslant h \leqslant p - 1$ such that $h^{(p-1)/q} \bmod p > 1$. α is a generator with order q in GF(p). $\{p, q, \alpha\}$ are the public values, $\{a_i, i = 0, \ldots, t-1\}$ are the secret values.

It should be pointed out that according to Lemma 1 in Reference 20, if α is a generator with order q in GF(p), then $\alpha^r \bmod p = \alpha^{r \bmod q} \bmod p$, for any nonnegative integer r.

4.1 Group secret key and secret shadows generation phase

The group secret key is determined by KAC as $f(0)$. The secret shadow for each group member is also determined by KAC as

$$f(x_i) \bmod q \quad \text{for } i = 1, 2, \ldots, n$$

where x_i is the public value associated with each group member. The KAC also needs to compute one group public key y as

$$y = \alpha^{f(0)} \bmod p$$

for group signature verification purpose and public keys y_i as

$$y_i = \alpha^{f(x_i)} \bmod p, \quad \text{for } i = 1, 2, \ldots, n$$

for all group members.

4.2 (t, n) Threshold signature generation phase

This scheme allows any t group members to represent the group to sign a message m. Without losing generality, assume that the t group members involved can be denoted as $u_1, u_2, \ldots, u_t$. This phase can be further divided into two parts.

Part 1: Individual signature generation and verification. Members can sign the message simultaneously. Here, just the procedures associated with member u_i are described.

Member u_i randomly selects an integer, $k_i \in [1, q-1]$, and computes a public value, r_i, as

$$r_i = \alpha^{k_i} \bmod p$$

and makes r_i publicly available through a broadcast channel. Once all r_i are available, each member computes the product, r, as

$$r = \prod_{i=1}^{t} r_i \bmod p$$

Member u_i uses his secret keys, $f(x_i)$ and k_i, to sign the message m based on the modified signature scheme. Member u_i then solves the equation

$$s_i = f(x_i) \times m' \times \left(\prod_{j=1, j \neq i}^{t} \frac{-x_j}{x_i - x_j} \right) - k_i \times r \bmod q$$

for integer s_i, where $0 \leqslant s_i \leqslant q - 1$ and $m' = f(m)$, and transmits $\{m, s_i\}$ to a designated clerk. Note here that the individual signature, $\{r_i, s_i\}$, is a partial signature of message m.

Once the clerk receives the individual signature $\{r_i, s_i\}$

from u_i, he needs to verify the validity of this partial signature. The clerk uses u_i's public keys, x_i and y_i, and partial signature $\{r_i, s_i\}$ to compute

$$y_i^{m' \prod_{j=1, j\neq i}^{t} \frac{-x_j}{x_i - x_j}} = r_i^r \alpha^{s_i} \bmod p$$

where $m' = f(m)$. If the equation holds true, the partial signature $\{r_i, s_i\}$ of message m received from u_i is valid.

Part 2: (t, n) Signature generation. Once t partial signatures are received and verified by the clerk, the group signature of message m can be generated as $\{r, s\}$, where $s = s_1 + s_2 + \cdots + s_t \bmod q$.

4.3 (t, n) Threshold signature verification phase

After receiving the group signature $\{r, s\}$ of the message m, an outsider needs to use the group public key y to verify the validity of the signature. The verification procedure is given as

$$y^{m'} = r^r \alpha^s \bmod p$$

where $m' = f(m)$. If the equation holds true, the group signature $\{r, s\}$ is valid.

Theorem: If $y^{m'} = r^r \alpha^s \bmod p$, the group signature $\{r, s\}$ has been verified.

Proof: With the knowledge of secret shadow $f(x_i)$, user u_i is able to generate its partial signature $\{r_i, s_i\}$ for message m to satisfy

$$y_i^{m' \prod_{j=1, j\neq i}^{t} \frac{-x_j}{x_i - x_j}} = r_i^r \alpha^{s_i}$$

Multiplying the above equation for $i = 1, 2, \ldots, t$ gives

$$\prod_{i=1}^{t} y_i^{m' \prod_{j=1, j\neq i}^{t} \frac{-x_j}{x_i - x_j}} = \prod_{i=1}^{t} r_i^r \alpha^{s_i} \bmod p \tag{3}$$

With the knowledge of t pairs of $(x_i, f(x_i))$, the unique $(t-1)$th degree polynomial, $f(x)$, can be determined as

$$f(x) = \sum_{i=1}^{t} f(x_i) \prod_{j=1, j\neq i}^{t} \frac{x - x_j}{x_i - x_j} \bmod q$$

The left-hand side of eqn. 3 can be rewritten as

$$\alpha^{m' \sum_{i=1}^{t} f(x_i) \prod_{j=1, j\neq i}^{t} \frac{-x_j}{x_i - x_j} \bmod q} \bmod p$$
$$= \alpha^{m' f(0)} \bmod p$$
$$= y^{m'} \bmod p$$

Since the group signature $\{r, s\}$ can be expressed as

$$r = \prod_{i=1}^{t} r_i \bmod p \quad \text{and} \quad s = s_1 + s_2 + \cdots + s_t \bmod q$$

The right-hand side of eqn. 3 can be rewritten as

$$\left(\prod_{i=1}^{t} r_i\right)^r \alpha^{\sum_{i=1}^{t} s_i \bmod q} \bmod p$$
$$= r^r \alpha^s \bmod p$$

4.4 Security analysis

Here, several possible attacks are proposed, but none can successfully break the scheme.

(i) Derivation of the group secret key $f(0)$, and the secret shadows $f(x_i)$ for $i = 1, 2, \ldots, n$, from the group public key, $y = \alpha^{f(0)} \bmod p$, and the public keys for members, $y_i = \alpha^{f(x_i)} \bmod p$, for $i = 1, 2, \ldots, n$, are equivalent to solving the discrete logarithm problems.

(ii) Derivation of the secret shadow $f(x_i)$, from one or multiple partial signature pairs (r_i, s_i) based on the equation

$$s_i = f(x_i) \times m' \times \left(\prod_{j=1, j\neq i}^{t} \frac{-x_j}{x_i - x_j}\right) - k_i \times r \bmod q$$

has the same difficulty as the modified ElGamal signature scheme.

(iii) Derivation of the group secret key, $f(0)$, from one or multiple group signature pairs (r, s), based on the equation

$$s = f(0) \times m' - k \times r \bmod q$$

has the same difficulty as the modified ElGamal signature scheme.

(iv) An attacker might try to impersonate member u_i by randomly selecting an integer $k_i' \in [1, q-1]$ and broadcasting $r_i' = \alpha^{k_i} \bmod p$. Since the productive value,

$$r' = \left(\prod_{j=1, j\neq i}^{t} r_j\right) r_i' \bmod p$$

is determined by all t members, without knowing the secret shadow $f(x_i)$, the attacker cannot generate a valid partial signature pair (r_i', s_i'), to satisfy the verification equation as

$$y_i^{nm' \prod_{j=1, j\neq i}^{t} \frac{-x_j}{x_i - x_j}} = r_i'^{r'} \alpha^{s_i'} \bmod p$$

5 (t, n) Threshold signature scheme without the assistance of a mutually trusted party

This scheme is the combination of the two previous schemes. The group secret and public keys are determined by all group members according to the (n, n) signature scheme above. Since there is no mutually trusted party, each member acts as one KAC to generate and distribute his secret key to other members according to the (t, n) scheme.

There are some public parameters that should be agreed to by all group members:

(i) p, a large prime modulus, where $2^{511} < p < 2^{512}$,

(ii) q, a prime divisor of $p-1$, where $2^{159} < q < 2^{160}$,

(iii) α, where $\alpha = h^{(p-1)/q} \bmod p$, h is a random integer with $1 \leqslant h \leqslant p-1$ such that $h^{(p-1)/q} \bmod p > 1$.

5.1 Public keys generating phase

Each member randomly selects integers, z_i and x_i from $[1, p-1]$ and computes a corresponding public key as

$$y_i = \alpha^{z_i} \bmod p$$

$\{x_i, y_i\}$ are the member's public keys and $\{z_i\}$ is the member's secret key. Then, the group public key y is determined by all members as

$$y = \prod_{i=1}^{n} y_i \bmod p$$

Since there is no mutually trusted party, each member acts as one KAC to use the $(t, n-1)$ secret sharing scheme as we described in the previous section to distribute his secret key to the other $n-1$ members. Assuming u_i with the secret key z_i, u_i randomly selects a $(t-1)$th degree polynomial, $f_i(x)$, with $f_i(0) = z_i \bmod q$ and computes the secret shadow, $f_i(x_j) \bmod q$, and the public key, $y_{i,j} = \alpha^{f_i(x_j)} \bmod p$, for each member u_j.

5.2 (t, n) Threshold signature generation phase

Without losing generality, assume that the group members involved can be denoted as $u_1, u_2, \ldots, u_t$. This phase can be further divided into two parts.

Part 1: Individual signature generation and verification. Members can sign the message simultaneously. Here, just the procedures associated with member u_i are described.

Member u_i randomly selects an integer, $k_i \in [1, q-1]$, and computes a public value r_i as

$$r_i = \alpha^{k_i} \bmod p$$

and makes r_i publicly available through a broadcast channel. Once all r_i are available, each member computes the productive value r as

$$r = \left(\prod_{i=1}^{t} r_i \right) \bmod p$$

Member u_i uses his secret keys, z_i and k_i, and secret shadows, $f_j(x_i)$, for $j = t+1, t+2, \ldots, n$, to sign the message m based on the modified signature scheme and solves the equation

$$s_i = \left\{ z_i + \sum_{j=t+1}^{n} f_j(x_i) \times \left(\prod_{k=1, k \neq i}^{t} \frac{-x_k}{x_i - x_k} \right) \right\} \times m' - k_i \times r \bmod q$$

for integer s_i, where $0 \leqslant s_i \leqslant q-1$ and $m' = f(m)$, and transmits $\{m, s_i\}$ to a designated clerk. Note here that the individual signature, $\{r_i, s_i\}$, is a partial signature of message m.

Once the clerk receives the individual signature $\{r_i, s_i\}$ from u_i, he needs to verify the validity of this partial signature. The clerk uses u_i's public keys, x_i, y_i, and $y_{j,i}$, for $j = t+1, t+2, \ldots, n$, and partial signature $\{r_i, s_i\}$ to compute

$$\left\{ y_i \left(\prod_{j=t+1}^{n} y_{j,i} \right)^{\prod_{k=1, k \neq i}^{t} \frac{-x_k}{x_i - x_k}} \right\}^{m'} = r_i^{r_i} \alpha^{s_i} \bmod p$$

where $m' = f(m)$. If the equation holds true, the partial signature $\{r_i, s_i\}$ of message m received from u_i has been verified.

Part 2: (t, n) Signature generation. Once t partial signatures are received and verified by the clerk, the group signature of message m can be generated as $\{r, s\}$, where $s = s_1 + s_2 + \cdots + s_t \bmod q$.

5.3 (t, n) Threshold signature verification phase

The verification procedure is given as

$$y^{m'} = r^r \alpha^s \bmod p$$

where $m' = f(m)$. If the equation holds, the group signature $\{r, s\}$ is valid.

Theorem: If $y^{m'} = r^r \alpha^s \bmod p$, the group signature $\{r, s\}$ has been verified.

Proof: The proof is similar to the proof in the previous section.

One additional problem needs to be solved because these group members are not mutually trusted. The problem is how to convince the rest of the members that the secret shadows received from the dealer (one of the group users) are derived consistently from the same secret without revealing the secret to the others. The application of this problem is very important. For example, a dishonest member can cheat some members by giving them fake shadows. The communication errors (i.e. noise) can also result in fake shadows.

It is now shown that, with this scheme, it is very easy to prevent the dealer from cheating the others.

Theorem: Any received fake shadow can be easily detected by any member.

Proof: First, the situation of fake shadows caused by communication noise is examined. Member u_j receives a fake shadow, $f'_i(x_j)$, from the member u_i, and the corresponding public key is $y_{i,j} = \alpha^{f_i(x_j)} \bmod p$. Obviously, this fake shadow can be easily detected by u_j. Now consider what will happen if a dishonest member u_i picks up a fake shadow, $f'_i(x_j)$, and publishes the corresponding public key as $y_{i,j} = \alpha^{f'_i(x_j)} \bmod p$. Since it is known that if the member is honest and picks up all the real shadows, then with the knowledge of any t shadows from the rest of $n-1$ shadows, the same polynomial $f_i(x)$ can be reconstructed. There are C_t^{n-1} ways to reconstruct $f_i(x)$. In other words, all these reconstructed polynomials will pass through $f_i(0) = z_i$. However, if there are fake shadows, some reconstructed polynomials will be different from $f_i(x)$ and will not pass through $f_i(0) = z_i$ and, without knowing these shadows, this condition can still be examined by just knowing the public keys of these shadows. Using the theorem in the previous section, for any t public keys of secret shadows, y_{i,j_k}, for $k = 1, 2, \ldots, t$, we have

$$y_i = y_{i,j_1}^{\prod_{k=1, k \neq 1}^{t} \frac{-x_{j_k}}{x_{j_1} - x_{j_k}}} y_{i,j_2}^{\prod_{k=1, k \neq 2}^{t} \frac{-x_{j_k}}{x_{j_2} - x_{j_k}}} \cdots y_{i,j_t}^{\prod_{k=1, k \neq t}^{t} \frac{-x_{j_k}}{x_{j_t} - x_{j_k}}} \bmod p$$

Since any t out of n combination of the public values satisfies the above relation, members can verify their secret shadows individually.

The security analysis of this scheme is almost the same as the previous one. However, this scheme does not need the assistance of a mutually trusted party.

6 Conclusion

Three threshold digital signature schemes based on the difficulty of solving the discrete logarithm problem are proposed. The group signature can be generated when the number of participating members is larger than or equal to the threshold value. The size of the group signature and the verification time of the signature are the same as that of an individual signature. The first scheme is a special case which requires all group members to sign the message together. This scheme can be easily applied to generate digital multisignature. The second scheme provides a general solution and it requires the assistance of a mutually trusted party; however, the third scheme does not require the mutually trusted party.

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